

考試規範：

- ★ 本考試卷共七大題 (共二頁)，配分請見各題，合計 110 分。
- ★ 本考試禁止翻閱書籍、筆記，亦不得查詢任何資料。
- ★ 禁止使用電子設備 (如手機、計算機、電腦) 進行計算或查詢答案。
- ★ 所有解答須附詳細計算過程，否則不予計分。

1. Apply the L'Hôpital rule to evaluate the following limits.

(a) (6 pts) $\lim_{x \rightarrow 0^+} x^2 \ln x$

(b) (6 pts) $\lim_{k \rightarrow \infty} \left(1 + \frac{\pi}{k}\right)^k$

2. Evaluate the following integrals.

(a) (8 pts) $\int \left(x + \sqrt{x} + 1 + \frac{1}{\sqrt{x}} + \frac{1}{x}\right) dx$

(b) (8 pts) $\int (x^2 \sin x + \cos^2 x) dx$

(c) (8 pts) $\int (\tan^2 x + \tan^4 x) dx$

(d) (8 pts) $\int \frac{2x}{\sqrt{1+x^4}} dx$

(e) (8 pts) $\int \frac{e^x}{e^{2x} - 3e^x + 2} dx$

(f) (8 pts) $\int_{-1}^1 (x^3 + 3)\sqrt{4-x^2} dx$

3. Let a be a real number and $a > 0$.

(a) (8 pts) Evaluate $\int x \ln x dx$.

(b) (6 pts) If a satisfies the equation $\int_0^a x \ln x dx = 0$, find the value of a .

4. (8 pts) Find the average value of $g(x) = |x| - 1$ on the interval $[-1, 3]$.

5. (8 pts) Let f be continuous on its domain. Suppose that

$$\int_0^{x^3} f(3t^2 - 6t) dt = \ln x$$

Find the value of $f(2025)$. (★ Hint: $2025 = 27 \times 75$)

(請翻頁繼續作答 Please turn the page to continue answering)

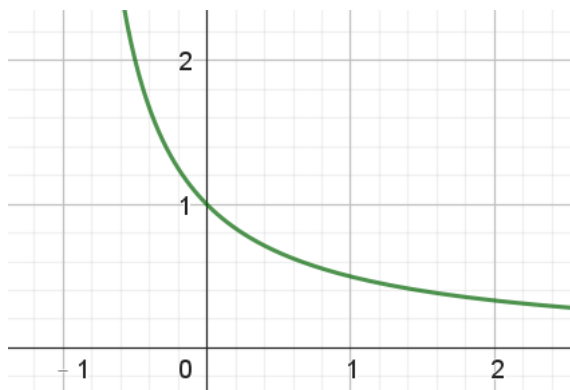
6. Let the function $f(x) = \frac{1}{1+x}$ be defined on the interval $[0, 2]$. Now, **divide** (分割) the interval $[0, 2]$ into n **equal sub-intervals** (n 等份的子區間). For **each** (每個) $k = 1, 2, \dots, n$, let the k -th sub-interval be

$$\left[\frac{2(k-1)}{n}, \frac{2k}{n} \right]$$

and **choose an arbitrary point** c_k **within the k -th sub-interval** (在第 k 個子區間中任意選擇一個點 c_k). Also, define the **sequence** S_n (數列 S_n) as

$$S_n = \frac{2}{n} [f(c_1) + f(c_2) + \dots + f(c_n)].$$

- (a) (3 pts) Is $S_1 > S_2$ always true? Give your reason!
- (b) (3 pts) What is the maximum value of S_4 ?
- (c) (4 pts) Find the value of $\lim_{n \rightarrow \infty} S_n$.



(The above figure shows the graph of the function $f(x) = 1/(1+x)$)

7. Let f be a function such that f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$.
- (a) (4 pts) Suppose f is one-to-one on $[a, b]$, prove that

$$\int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(x) dx = bf(b) - af(a)$$

(★ Hint: Let $y = f^{-1}(x)$, then $y = f^{-1}(x) \iff x = f(y)$)

- (b) (6 pts) Evaluate the integral $\int_{11}^{30} \sqrt{\sqrt[3]{x-3}-2} dx$.

(★ Hint: Note that $f(x) = \sqrt{\sqrt[3]{x-3}-2}$ is one-to-one on $[11, 30]$)

(考題結束 End of final exam questions)