

注意事項：無計算過程或說明不給分，總分 100 分。

1. Suppose that the vector function $\mathbf{r}(t)$ is the position of a particle at time t satisfying

$$\frac{d^2\mathbf{r}}{dt^2} = e^t \mathbf{i} - e^{-t} \mathbf{j} + 4e^{2t} \mathbf{k},$$

$$\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{r}'(0) = -\mathbf{i} + 4\mathbf{j}.$$

- (a) (5 points) Find the particle's velocity $\mathbf{v}(t)$ at time t .
 (b) (5 points) Find the particle's position $\mathbf{r}(t)$ at time t .
2. (10 points) Let f be a real-valued function defined on an open interval I containing x . If there exists a constant $M > 0$ such that the fourth derivative satisfies $|f^{(4)}(z)| \leq M$ for all $z \in I$, use the Taylor's Theorem to prove that the following inequality

$$\left| f''(x) - \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \right| \leq \frac{Mh^2}{12}$$

holds for all $h \neq 0$ with $x-h, x+h \in I$.

3. Let C be the polar curve $r = \cos^2(\theta/2)$ for $0 \leq \theta \leq 2\pi$.

- (a) (10 points) Find the area of the polar region enclosed by C .
 (b) (10 points) Find the length of the curve C from $\theta = 0$ to $\theta = \pi$.

4. (32 points) Determine whether the series converges or diverges. Give your reasons.

(a) $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)}$ (b) $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$ (c) $\sum_{n=1}^{\infty} \frac{2^n \cos n}{(-3)^n}$ (d) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$

5. (8 points) Find the sum of the telescoping series $\sum_{n=1}^{\infty} \frac{2}{4n^2 + 8n + 3}$.

6. (10 points) If x and y are defined by the parametric equations

$$x = -5 + 3t^2 \quad \text{and} \quad y = \tan^{-1} t,$$

show that the second derivative $\frac{d^2y}{dx^2}$ is positive for all $t < 0$.

7. (10 points) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n (2x+5)^n}{\sqrt{2n+5}}$. For what values of x does the series converge conditionally?