

國立臺灣師範大學 114 學年度暑期/115 年度先修 微積分乙 (一) 期中考

考試注意事項：

- ★ 本試卷共八大題 (共二頁)，配分請見各題，合計 120 分。
- ★ 本考試禁止帶手機、計算機及電腦等電子用品進入考場。
- ★ 不必抄題，可用中文作答。作答時請於答案本上標示題號。
- ★ 除了是非題、填充題和多選題可直接寫答案外，其餘需要有詳細過程。

1. (是非題, 1×8 分) If the statement is true, answer $\bigcirc$, otherwise, answer $\times$.

- (a) For any $x \in \mathbb{R}$, we have $e^{\ln x} = x$.
- (b) For any $x \in \mathbb{R}$, we have $\sin^2 x = \frac{1 - \cos(2x)}{2}$ and $\cos^2 x = \frac{1 + \cos(2x)}{2}$.
- (c) A function f is said to be strictly increasing on an interval I if $f(a) < f(b)$ whenever $a, b \in I$ with $a < b$.
- (d) A differentiable function must be a continuous function.
- (e) Let g be a function defined on the interval $[a, b]$. If $g(a) \cdot g(b) < 0$, then there is a number $c \in (a, b)$ such that $g(c) = 0$.
- (f) If the function h is continuous on (a, b) , then h has both absolute maximum and absolute minimum on (a, b) .
- (g) Note that $\lim_{x \rightarrow 2026} \frac{2026}{(x - 2026)^{2026}} = \infty$
- (h) If $\lim_{x \rightarrow c} f(x)$ exists but $\lim_{x \rightarrow c} g(x)$ does not exist, then $\lim_{x \rightarrow c} [f(x) + g(x)]$ does not exist.

2. (計算題, 5×5 分) Find the limit if it exists. Notice that L'Hôpital's Rule is forbidden.

- (a) $\lim_{\theta \rightarrow 0} \cot(4\theta) \tan(5\theta)$
- (b) $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{12}{x^3-8} \right)$
- (c) $\lim_{t \rightarrow 0} \frac{2t + \sin t}{2t - \cos t}$
- (d) $\lim_{x \rightarrow 0} \frac{\ln(1-3x)}{x}$
- (e) $\lim_{x \rightarrow \infty} e^{-2x} \cos(2x)$

3. (計算題, 5×6 分) Derivatives questions

- (a) Given $y = \frac{1}{1-x^{-1}}$. Find the derivative $\frac{dy}{dx}$.
- (b) Given $y = (114)^{\sqrt{\sin x}} + (\sin x)^{\sqrt{114}}$. Find the derivative $\frac{dy}{dx}$.
- (c) Given $y = x^3 \log_3(x^3 + 3) + e^3$. Find the derivative $\frac{dy}{dx}$.
- (d) Given a function $f(x) = (\sec x)^{\tan x}$. Find the value of $f'\left(\frac{\pi}{4}\right)$.
- (e) Given a function $f(x) = \cos(\tan^{-1}(x))$. Find the value of $f'(2)$.
(★ Please simplify your answer and exclude trigonometric function values.)
- (f) Let $f(x)$ be a differentiable function satisfying

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 3, \quad f'(0) = 10, \quad f'(1) = 6, \quad f'(2) = 1$$

If $F(x) = f\left(x + f(x + f(x))\right)$, find the value of $F'(0)$.

(請翻頁繼續作答 Please turn the page to continue answering)

4. (計算題, 7 分) Find an equation of the tangent line to the graph of

$$\sin(x+y) = y^2 \cos x$$

at the point $(0, 0)$.

5. (說明題, 10 分) Find the constant a and b such that the following function f is differentiable at $x = 0$.

$$f(x) = \begin{cases} \sin(2x) & , x \leq 0 \\ 2x^2 + ax + b - 3 & , x > 0 \end{cases}$$

6. Consider the function $h(x) = \frac{x^2 - 1}{x^4 - 1}$.

- (a) (填充題, 2 分) The domain of $h(x)$ is $\text{dom}(h) = ?$
- (b) (多選題, 5 分) Which of the following statements **are** true?
- (A) The function $h(x)$ is a rational function.
- (B) Since $\lim_{x \rightarrow \pm\infty} \frac{x^2 - 1}{x^4 - 1} = 0$, h has no horizontal asymptote.
- (C) For any $c_1 \in \text{dom}(h)$, the line $x = c_1$ is **not** a vertical asymptote for h .
- (D) For any $c_2 \notin \text{dom}(h)$, the line $x = c_2$ is a vertical asymptote for h .
- (E) For any $c_2 \notin \text{dom}(h)$, we can appropriately (適當地) define $h(c_2)$ such that the function h becomes continuous at $x = c_2$.
7. Let f be a **twice differentiable** (二次可微分, 即 $f''(x)$ 存在), one-to-one function on $\mathbb{R}$ that satisfies

$$[f'(x)]^2 - [f(x)]^2 = 1 \quad \forall x \in \mathbb{R} \quad \text{and} \quad f'(x) > 0 \quad \forall x \in \mathbb{R}$$

Suppose that the limit $\lim_{x \rightarrow 0} \frac{f(x)}{x}$ exists.

- (a) (計算題, 3 分) Find the value of $f(0)$.
- (b) (證明題, 5 分) Show that $f''(x) = f(x)$ for all $x \in \mathbb{R}$.
- (c) (說明題, 4 分) Determine the open intervals on which f is concave upward (凹向上) or concave downward (凹向下).
- (d) (說明題, 4 分) Find the coordinates (坐標) of the point on the graph of $y = f(x)$ where the slope of the tangent line is minimized.
- (e) (證明題, 4 分) Show that $\frac{d}{dx} [f^{-1}(x)] = \frac{1}{\sqrt{1+x^2}}$.
8. Applications.
- (a) (敘述題, 3 分) Please state the Mean Value Theorem (請敘述均值定理).
- (b) (證明題, 5 分) Show that $|\cos a - \cos b| \leq |a - b|$ for all a and b .
(★ Hint: Please discuss the cases for $a = b$ and $a \neq b$.)
- (c) (計算題, 5 分) Use the fact of (b) to find the limit $\lim_{x \rightarrow \infty} [\cos(\sqrt{x^2 + 4}) - \cos(\sqrt{x^2 - 5})]$.

(考題結束 End of midterm exam questions)