

國立臺灣師範大學 114 學年度暑期/115 年度先修 微積分乙 (一) 期末考

考試注意事項：

- ★ 本試卷共八大題 (共二頁)，配分請見各題，合計 120 分。
- ★ 本考試禁止帶手機、計算機及電腦等電子用品進入考場。
- ★ 不必抄題，可用中文作答。作答時請於答案本上標示題號。
- ★ 除了是非題和填充題可直接寫答案外，其餘需要有詳細過程。

1. (是非題, 1×8 分) If the statement is true, answer $\bigcirc$, otherwise, answer $\times$.

- (a) All continuous functions have derivatives.
- (b) All continuous functions have anti-derivatives.
- (c) Note that

$$\lim_{t \rightarrow 0} \frac{2t - \sin t}{2t - \cos t} \stackrel{L.}{=} \lim_{t \rightarrow 0} \frac{2 - \cos t}{2 + \sin t} \stackrel{L.}{=} \lim_{t \rightarrow 0} \frac{\sin t}{\cos t} = \lim_{t \rightarrow 0} (\tan t) = 0$$

- (d) Note that

$$\lim_{x \rightarrow \infty} x \tan \left(\frac{1}{x} \right) = \lim_{x \rightarrow \infty} \frac{\tan \left(\frac{1}{x} \right)}{\frac{1}{x}} \stackrel{L.}{=} \lim_{x \rightarrow \infty} \frac{\sec^2 \left(\frac{1}{x} \right) \cdot \left(-\frac{1}{x^2} \right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \sec^2 \left(\frac{1}{x} \right) = (\sec 0)^2 = 1$$

- (e) Note that

$$\int_{-2}^3 \frac{1}{x+1} dx = \ln |x+1| \Big|_{-2}^3 = \ln 4 - \ln 1 = \ln 4$$

- (f) The area bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ is

$$A = \int_a^b [f(x) - g(x)] dx$$

- (g) If the function f is continuous on $[1, \infty)$ and $\lim_{x \rightarrow \infty} f(x) = 0$, then the integral

$$\int_1^{\infty} f(x) dx \text{ is convergent (收斂的).}$$

- (h) If f is continuous at $[a, b]$, then there exists $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$

2. (填充題, 2×3 分) Evaluate the following.

$$(a) \int_0^1 \frac{d}{dx} [e^{x^2}] dx \quad (b) \frac{d}{dx} \left[\int_0^1 e^{x^2} dx \right] \quad (c) \frac{d}{dx} \left[\int_0^x e^{t^2} dt \right]$$

3. (計算題, 6×2 分) Apply L'Hôpital's Rule at least once to evaluate the following limits.

- (a) $\lim_{x \rightarrow 0^+} (\csc x - \cot x)$
- (b) $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^{bx}$, where a, b are real numbers.

(請翻頁繼續作答 Please turn the page to continue answering)

4. (計算題, 6×4 分) Evaluate the following integrals.

(a) $\int_2^5 (3 - |3 - x|) dx$

(b) $\int \frac{1}{\theta^2} \cot^2 \left(\frac{1}{\theta} \right) d\theta$

(c) $\int \frac{1}{\sqrt{2x - x^2}} dx$

(d) $\int \frac{1}{1 - e^x} dx$

5. (計算題, 7×4 分) Evaluate the following integrals.

(a) $\int_1^e \frac{\ln x}{x\sqrt{1 + (\ln x)^2}} dx$

(b) $\int 2x^3 \sin(x^2) dx$

(c) $\int (\cos^2 x - \cos^4 x) dx$

(d) $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

6. Applications.

(a) (計算題, 8 分) Evaluate the improper integral $\int_0^1 \frac{\ln x}{4\sqrt{x}} dx$.

(b) (計算題, 6 分) Find the value of $\lim_{x \rightarrow 1} \frac{1 + \int_0^{x^2} \frac{\ln t}{4\sqrt{t}} dt}{\sqrt{x} \ln x - 2\sqrt{x} + 2}$

7. Suppose that f is a **differentiable function** and **one-to-one function** on $\mathbb{R}$ that satisfies

$$[f'(x)]^2 - [f(x)]^2 = 1 \quad \forall x \in \mathbb{R} \quad \text{and} \quad f'(x) > 0 \quad \forall x \in \mathbb{R}$$

Let the limit $\lim_{x \rightarrow 0} \frac{f(x)}{x}$ exist and its value be equal to a constant L .

(a) (計算題, 4 分) Find the value of $f^{-1}(0)$.

(b) (證明題, 4 分) Show that $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{\sqrt{1 + x^2}}$.

(c) (計算題, 7 分) Use the information from (a) and (b) to find $f^{-1}(x)$.

(d) (計算題, 5 分) Find $f(x)$.

8. (計算題, 8 分) Given a function f satisfies the following things:

$$f(0) = 1, \quad f'(0) = 2 \quad \text{and} \quad \int_0^\pi [f(x) + f''(x)] \sin x dx = 3$$

Find the value of $f(\pi)$.

(考題結束 End of final exam questions)