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Study on the solvability of solutions for double-degeneracy problems by using the indirect BIEM/BEM

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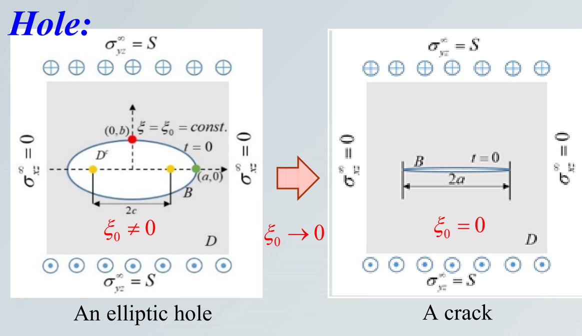
Abstract

In this study, the indirect boundary element method (BEM) of the single and double layers can be employed to solve the anti-plane shear problem including the crack or rigid-line inclusion. The exact solution and numerical results for the unknown boundary density and displacement field has been solved. We discuss the solvability of solutions for four different problems by using the indirect BIEM, respectively. It is interesting to find that even the unknown boundary data does not agree with the analytical solution, the solution for the displacement field is still acceptable, for the single degeneracy of degenerate boundary (SDB) cases. The reason is explained by applying the singular value decomposition (SVD) technique. In addition, we deal with the double-degeneracy problem by using the Fichera method. It is noteworthy that the Fichera method can effectively solve the degenerate-scale problem.

Problem statement

Governing equation $\nabla^2 u(\mathbf{x}) = 0,\ \mathbf{x} \in D$ $\quad u(\mathbf{x})$: Displacement

Hole:

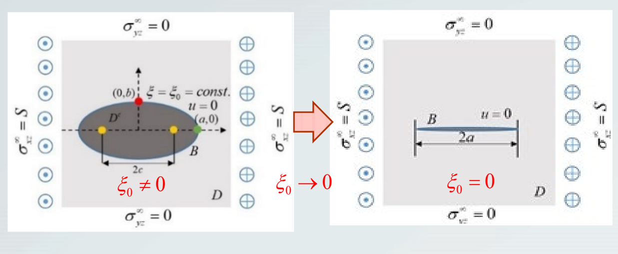


An elliptic hole $\qquad$ A crack

Remote anti-plane shear $\quad u^{\infty}(\mathbf{x}) = \frac{S}{\mu} y$, when $\sigma_{yz}^{\infty} = S$

Boundary condition $\quad t(\mathbf{x}) = \frac{\partial u(\mathbf{x})}{\partial n_x} = 0,\ \mathbf{x} \in B$

Rigid inclusion:



An elliptic rigid inclusion $\qquad$ A rigid line inclusion

Remote anti-plane shear $\quad u^{\infty}(\mathbf{x}) = \frac{S}{\mu} x$, when $\sigma_{xz}^{\infty} = S$

Boundary condition $\quad u(\mathbf{x}) = 0,\ \mathbf{x} \in B$

Indirect boundary integral formulations

Single-layer potential $\quad u(\mathbf{x}) = \int_B U(\mathbf{x},\mathbf{s})\alpha(\mathbf{s})dB,\ \mathbf{x} \in D \cdots$(A) $\xrightarrow{\partial/\partial n_x}$ $t(\mathbf{x}) = \int_B T(\mathbf{x},\mathbf{s})\alpha(\mathbf{s})dB,\ \mathbf{x} \in D \cdots$(B)

Double-layer potential $\quad u(\mathbf{x}) = \int_B U^*(\mathbf{x},\mathbf{s})\beta(\mathbf{s})dB,\ \mathbf{x} \in D \cdots$(C) $\xrightarrow{\partial/\partial n_x}$ $t(\mathbf{x}) = \int_B T^*(\mathbf{x},\mathbf{s})\beta(\mathbf{s})dB,\ \mathbf{x} \in D \cdots$(D)

Fundamental solution $\quad U(\mathbf{x},\mathbf{s}) = \ln r,\ r = |x - s|$

Kernel functions $\quad T(\mathbf{x},\mathbf{s}) = \frac{\partial U(\mathbf{x},\mathbf{s})}{\partial \mathbf{n}_x},\ U^*(\mathbf{x},\mathbf{s}) = \frac{\partial U(\mathbf{x},\mathbf{s})}{\partial \mathbf{n}_s},\ T^*(\mathbf{x},\mathbf{s}) = \frac{\partial^2 U(\mathbf{x},\mathbf{s})}{\partial \mathbf{n}_x \partial \mathbf{n}_s}.$

Unknown boundary densities $\quad \alpha(\mathbf{s}) = \frac{1}{J_s}\left(a_0 + \sum_{n=1}^{\infty} b_n \cos n\eta_s + \sum_{n=1}^{\infty} b_n \sin n\eta_s\right),\ 0 \le \eta_s < 2\pi,\ \mathbf{s} = (\xi_s,\eta_s) \in B$

$$\beta(\mathbf{s}) = c_0 + \sum_{n=1}^{\infty} d_n \cos n\eta_s + \sum_{n=1}^{\infty} d_n \sin n\eta_s,\ 0 \le \eta_s < 2\pi,\ \mathbf{s} = (\xi_s,\eta_s) \in B$$

Fichera formulation

$$\begin{cases} u^M(\mathbf{x}) = \int_B U(\mathbf{x},\mathbf{s})\alpha(\mathbf{s})\, dB(\mathbf{s}) + c \\ \int_B \alpha(\mathbf{s})\, dB(\mathbf{s}) = 0 \end{cases} \cdots (\mathrm{A}')$$

For degenerate scale of rigid line inclusion.

Kernel function $\quad U(\mathbf{x},\mathbf{s})$

Unknown boundary density

$$\alpha(\mathbf{s}) = \frac{1}{J_s}\left(a_0 + \sum_{n=1}^{\infty} b_n \cos n\eta_s + \sum_{n=1}^{\infty} b_n \sin n\eta_s\right)$$

Results and discussions

Table 1 Solution of crack and rigid-line inclusion problems by using the single and double-layer approaches (analytical and numerical approaches).

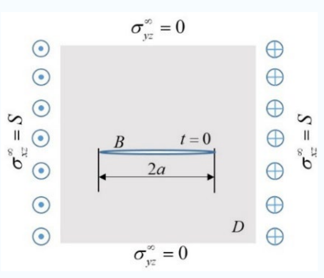
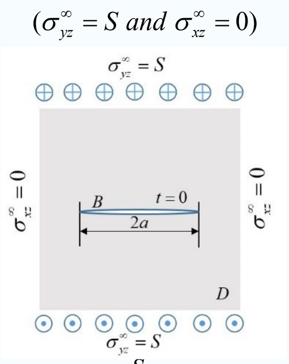
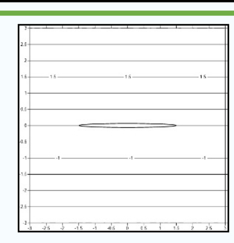
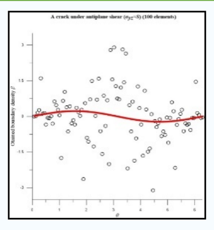
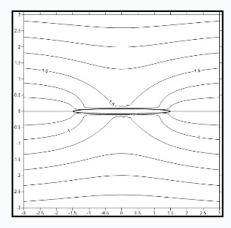
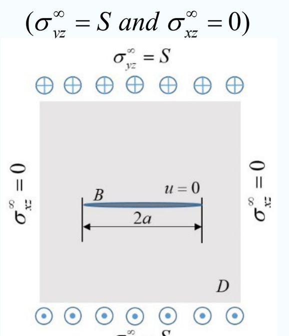
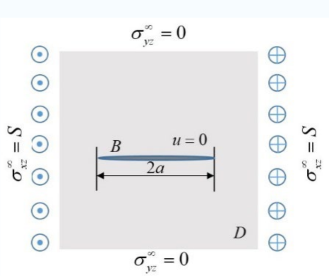
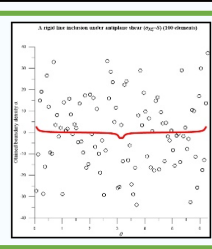
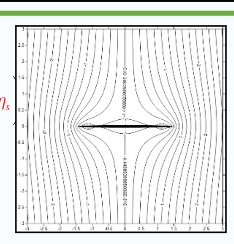
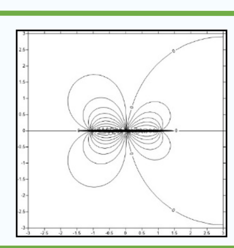
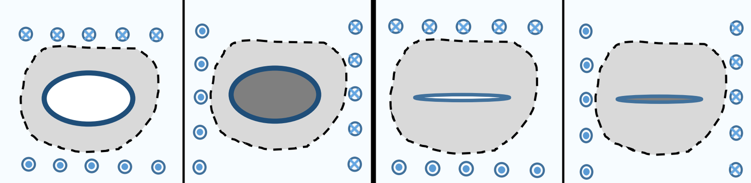
Trivial cases	Nontrivial cases	Approach	Boundary density		Field solution		
			Degenerate boundary, a=1.5, b=0, 100 elements				
Crack ($\sigma_{yz}^{\infty}=0$ and $\sigma_{xz}^{\infty}=S$)  $t^M(\xi_0,\eta_x) = \frac{S}{\mu}\frac{c}{J(\xi_0,\eta_x)}\cancel{\sinh\xi_0}\cos\eta_x$	**Crack** ($\sigma_{yz}^{\infty}=S$ and $\sigma_{xz}^{\infty}=0$)  $t^M(\xi_0,\eta_x) = \frac{S}{\mu}\frac{c}{J(\xi_0,\eta_x)}\sin\eta_x$	Single-layer approach	$a_0=0$; $a_n=0,\ n=1,2,3,\ldots$; b_1 has no solution; b_n is free, $n=2,3,4,\ldots$ ⇒ Since $\sinh\xi_x=0$ is in the denominator, there is no solution ⇒ Degenerate boundary	Fail	$\alpha(\mathbf{s})$ has no solution; $u_z(\mathbf{x})$ can not be found		X
		Double-layer approach	c_0 is free, c_n is free, $n=1,2,3,\ldots$; $d_1 = \frac{Sc}{2\pi\mu}$; $d_n=0,\ n=2,3,4,\ldots$ ⇒ The kernel in the Double-layer BIE has no constant ⇒ Degenerate boundary		$\beta(\mathbf{s}) = c_0 + \sum_{n=1}^{\infty} c_n \cos n\eta_s + \frac{Sc}{2\pi\mu}\sin\eta_s$; $u_z(\mathbf{x}) = \frac{Sc}{\mu}\cosh\xi_x \sin\eta_x$		O
Rigid-line inclusion ($\sigma_{yz}^{\infty}=S$ and $\sigma_{xz}^{\infty}=0$)  $u^M(\xi_0,\eta_x) = -\frac{Sc}{\mu}\cancel{\sinh\xi_x}\sin\eta_x$	**Rigid-line inclusion** ($\sigma_{yz}^{\infty}=0$ and $\sigma_{xz}^{\infty}=S$)  $u^M(\xi_0,\eta_x) = -\frac{Sc}{\mu}\cos\eta_x$	Single-layer approach	$\left(\ln\frac{c}{2}\right)p_0=0$; $p_1 = \frac{Sc}{2\pi\mu}$; $p_n=0,\ n=2,3,\ldots$; q_n is free, $n=1,2,3,4,\ldots$ ⇒ Degenerate scale $a=c=2$ ⇒ Degenerate boundary		$\alpha(\mathbf{s}) = \frac{1}{J(\xi_0,\eta_s)}\left(a_0 + \frac{Sc}{2\pi\mu}\cos\eta_s + \sum_{n=1}^{\infty} b_n \sin n\eta_s\right)$; $u_z(\mathbf{x}) = 2\pi\left(\xi_x + \ln\frac{c}{2}\right)a_0 + \frac{Sc}{\mu}\sinh\xi_x\cos\eta_x$		O
		Double-layer approach	p_0 is free; p_1 has no solution; p_n is free, $n=2,3,\ldots$; $q_n=0,\ n=1,2,3,4,\ldots$ ⇒ The kernel in the Double-layer BIE has no constant ⇒ Since $\sinh\xi_x=0$ is in the denominator, there is no solution ⇒ Degenerate boundary	Fail	$\beta(\mathbf{s})$ has no solution; $u_z(\mathbf{x})$ can not be found		X

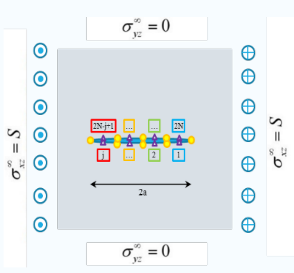
Table 2 Solvability of solutions for four anti-plane shear problems by using direct and indirect BIEMs.

Methods		Problems (Image)	Ordinary scale		Degenerate scale	
						
Direct method	Singular formula (UT formula)	Ordinary scale	√	√	X (SDB)	√ (SDB)
		Degenerate scale		X (SDS)		X (DD)
	Hyper-singular formula (LM formula)	Ordinary scale	√	√	√ (SDB)	X (SDB)
		Degenerate scale		√		X (DD)
Indirect method	Single-layer potential	Ordinary scale	√	√	X (SDB)	√ (SDB)
		Degenerate scale		X (SDS)		X (DD)
	Double-layer potential	Ordinary scale	√	√	√ (SDB)	X (SDB)
		Degenerate scale		√		X (DD)

√ : solvable X : unsolvable, SDS : Single degeneracy of degenerate scale, SDB : Single degeneracy of degenerate boundary, DD : Double degeneracy.

Example (SDB)

a=0.2, b=0, N=4, S=1, μM=1, 10 elements.



the ith collocation point

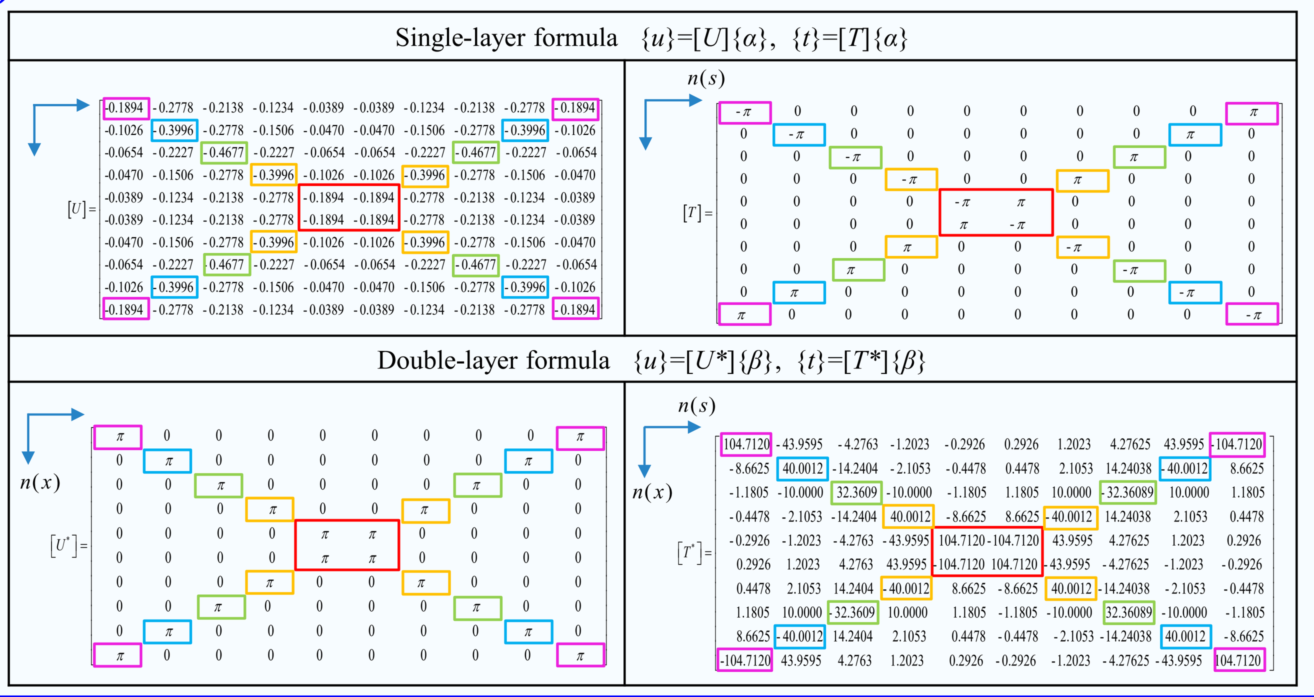
the jth constant element

the kth node

Table 3 The influence coefficient matrix.

Single-layer formula $\{u\}=[U]\{\alpha\},\ \{t\}=[T]\{\alpha\}$

Double-layer formula $\{u\}=[U^*]\{\beta\},\ \{t\}=[T^*]\{\beta\}$



Explanation of correct displacement field for the boundary density including the homogeneous solution

Discretization of BIE

$$[U_B]_{10\times10}\{\alpha_g\}_{10\times1} = \{u^M\}_{10\times1},$$
$$[U_D]_{M\times10}\{\alpha_g\}_{10\times1} = \{u_D\}_{M\times1}.$$

The solution of boundary density

$$\{\alpha\} = \{\alpha_g\} = \{\alpha_c\} + \{\alpha_p\}.$$

Thus

$$\{u^M\} = [U_B]\{\alpha_c\} + [U_B]\{\alpha_p\}$$
$$[U_B]\{\alpha_c\} = 0.$$

$[U_B]$: influence matrices of the **boundary**

$[U_D]$: influence matrices of the **domain**

$\{\alpha_c\}$: the complementary solution of the $\{\alpha\}$, contributed by the row space vector $\psi_i,\ i=1,2,\cdots,N$.

$\{\alpha_p\}$: the particular solution of the $\{\alpha\}$, contributed by the row space vector $\psi_i,\ i=N+1,\cdots,2N$.

$\{\alpha_g\}$: general solution of $\{\alpha\}$.

Based on SVD technique

$$[U_B]_{10\times10} = \left[\Phi^{(U_B)}\right]_{10\times10}\left[\Sigma^{(U_B)}\right]_{10\times10}\left[\Psi^{(U_B)}\right]^T_{10\times10},\quad [U_D]_{M\times10} = \left[\Phi^{(U_D)}\right]_{M\times M}\left[\Sigma^{(U_D)}\right]_{M\times10}\left[\Psi^{(U_D)}\right]^T_{10\times10},$$

we know

$$\{\alpha_c\} = \sum_{i=1}^{5} w_i^{\alpha}\left\{\psi_i^{(U_B)}\right\} \text{ and } \{\alpha_p\} = \sum_{i=6}^{10} w_i^{\alpha}\left\{\psi_i^{(U_B)}\right\},\ \text{where } w_i^{\alpha} = \left\{\psi_i^{(U_B)}\right\}\cdot\{\alpha\},\ i=1,2,\ldots,10.$$

Beside

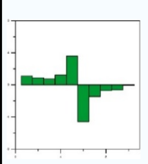
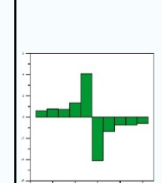
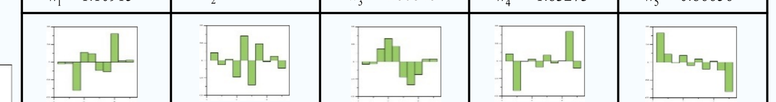
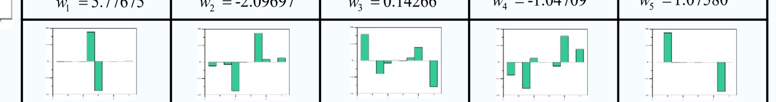
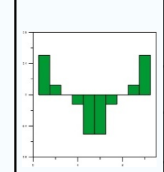
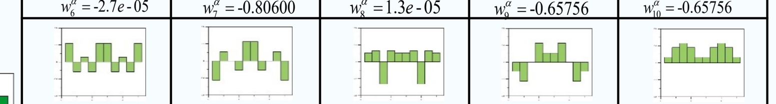
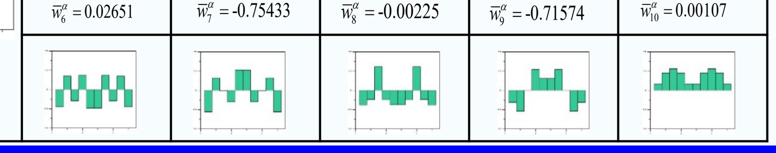
$$\psi_i^{(U_D)} = \sum_{i=1}^{5} c_i\left\{\psi_i^{(U_B)}\right\},\quad \{\alpha_c\} = \sum_{i=1}^{5} \overline{w}_i^{\alpha}\left\{\psi_i^{(U_D)}\right\},$$

Therefore

$$[U_D]\{\alpha_c\} = 0,$$
$$\{u_D\} = [U_D]\{\alpha_p\} + [U_D]\{\alpha_c\} = [U_D]\{\alpha_p\}.$$

We successfully prove that the homogeneous solution has no effect on the final displacement field. That is the reason why the unknown boundary data does not agree with the analytical solution, but the displacement field solution is still acceptable.

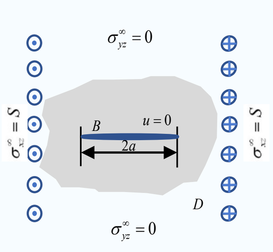
Table 4 Solutions of boundary densities including the homogeneous part.

α_g		$\psi_1^{(U_B)}$	$\psi_2^{(U_B)}$	$\psi_3^{(U_B)}$	$\psi_4^{(U_B)}$	$\psi_5^{(U_B)}$
	α_c 	$w_1^{\alpha}=1.10915$	$w_2^{\alpha}=3.66937$	$w_3^{\alpha}=4.60846$	$w_4^{\alpha}=-1.83273$	$w_5^{\alpha}=0.86656$
						
		$\psi_1^{(U_D)}$	$\psi_2^{(U_D)}$	$\psi_3^{(U_D)}$	$\psi_4^{(U_D)}$	$\psi_5^{(U_D)}$
		$\overline{w}_1^{\alpha}=5.77675$	$\overline{w}_2^{\alpha}=-2.09697$	$\overline{w}_3^{\alpha}=0.14266$	$\overline{w}_4^{\alpha}=-1.04709$	$\overline{w}_5^{\alpha}=1.07580$
						
	α_p	$\psi_6^{(U_B)}$	$\psi_7^{(U_B)}$	$\psi_8^{(U_B)}$	$\psi_9^{(U_B)}$	$\psi_{10}^{(U_B)}$
		$w_6^{\alpha}=-2.7e\text{-}05$	$w_7^{\alpha}=-0.80600$	$w_8^{\alpha}=1.3e\text{-}05$	$w_9^{\alpha}=-0.65756$	$w_{10}^{\alpha}=-0.65756$
						
		$\psi_6^{(U_D)}$	$\psi_7^{(U_D)}$	$\psi_8^{(U_D)}$	$\psi_9^{(U_D)}$	$\psi_{10}^{(U_D)}$
		$\overline{w}_6^{\alpha}=0.02651$	$\overline{w}_7^{\alpha}=-0.75433$	$\overline{w}_8^{\alpha}=-0.00225$	$\overline{w}_9^{\alpha}=-0.71574$	$\overline{w}_{10}^{\alpha}=0.00107$
						

Double degeneracy

(2N=10) (a=2.0685, b=0) (symmetric problem)

Rigid-line inclusion (10 elements) ➡ $r_a = N-1,\quad 2N - r_a = N+1$ ➡ The forcing vector $\{u^M\}$ is symmetric.



$[U_B]\{\alpha\} = \{u^M\}$

(single-layer approach)

$$[U_B]_{2N\times2N} = \left[\Phi^{(U_B)}\right]_{2N\times2N}\left[\Sigma^{(U_B)}\right]_{2N\times2N}\left[\Psi^{(U_B)}\right]^T_{2N\times2N}$$

Table 5 The action of the matrix $[U_B]$ and its spectrum in the double degeneracy.

the superscript "s" denotes symmetry
the superscript "a" denotes anti-symmetry

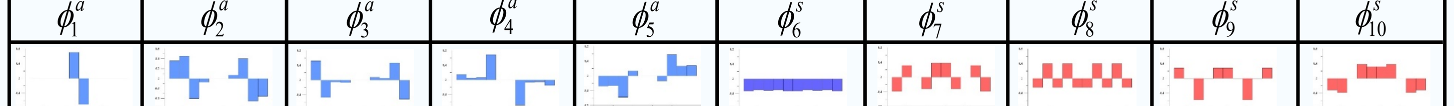
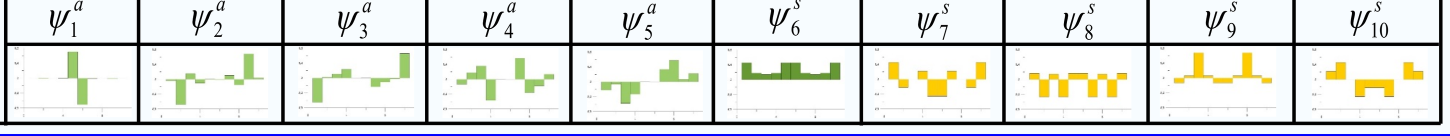
σ_i	$\sigma_i = 0, i=1,\ldots,6$					$\sigma_i \neq 0, i=7,\ldots,10$				
$u^M \cdot \phi$	$u^M \cdot \phi^a = 0$ (Solvable, Infinitely solution)					$u^M \cdot \phi^s \neq 0$ (for symmetry excitation)				
	ϕ_i^a : anti-symmetry ψ_i^a : anti-symmetry					ϕ_i^s : symmetry ψ_i^s : symmetry				
ϕ_i	ϕ_1^a	ϕ_2^a	ϕ_3^a	ϕ_4^a	ϕ_5^a	ϕ_6^s	ϕ_7^s	ϕ_8^s	ϕ_9^s	ϕ_{10}^s
										
ψ_i	ψ_1^a	ψ_2^a	ψ_3^a	ψ_4^a	ψ_5^a	ψ_6^s	ψ_7^s	ψ_8^s	ψ_9^s	ψ_{10}^s
										

Table 6 Solution of double-degeneracy case (Fichera Formulation).

$\nabla^2 u^{\infty}=0$; $\sigma_{yz}=0=\mu\frac{\partial u^{\infty}}{\partial y},\ \sigma_{xz}=S=\mu\frac{\partial u^{\infty}}{\partial x}$; $u^{\infty}(\xi_x,\eta_x)=\frac{Sc}{\mu}\cosh\xi_x\cos\eta_x$	An elliptical case	
	Single-layer Formulation	Fichera Formulation
Given BC	$u^M(x) = 0 - \frac{S}{\mu}c\cosh\xi_x\cos\eta_x$	
Unknown Boundary Density	$\alpha(s) = \frac{1}{J_s}\left(a_0 + \sum_{n=1}^{\infty} a_n\cos n\eta_s + b_n\sin n\eta_s\right)$	
Formulation	$u^M(\mathrm{x}) = \int_B U(\mathrm{x,s})\alpha(\mathrm{s})\,dB(\mathrm{s})$	$\begin{cases} u^M(\mathrm{x}) = \int_B U(\mathrm{x,s})\alpha(\mathrm{s})\,dB(\mathrm{s}) + C \\ \int_B \alpha(\mathrm{s})\,dB(\mathrm{s}) = 0 \end{cases}$
$U(\mathrm{x,s})$	$U^e(\xi_x,\eta_x;\xi_s,\eta_s) = \left(\xi_x + \ln\frac{c}{2} - \sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_x}\cosh m\xi_s\cos m\eta_x\cos m\eta_s - \sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_x}\sinh m\xi_s\sin m\eta_x\sin m\eta_s\right)$	
BIE for boundary point	$-\frac{S}{\mu}c\cosh\xi_0\cos\eta_x = \left(2\pi\left(\xi_0 + \ln\frac{c}{2}\right)a_0 - \sum_{n=1}^{\infty}\frac{2\pi}{n}e^{-n\xi_0}\cosh n\xi_0\cos n\eta_x a_n - \sum_{n=1}^{\infty}\frac{2\pi}{n}e^{-n\xi_0}\sinh n\xi_0\sin n\eta_x b_n\right)$	$\frac{S}{\mu}c\cosh\xi_0\cos\eta_x = \left(\sum_{n=1}^{\infty}\frac{2\pi}{n}e^{-n\xi_0}\cosh n\xi_0\cos n\eta_x a_n + \sum_{n=1}^{\infty}\frac{2\pi}{n}e^{-n\xi_0}\sinh n\xi_0\sin n\eta_x b_n\right)$
Unknown Fourier Coefficient & undetermined coefficient, C	$\begin{cases} \left(\xi_0 + \ln\frac{c}{2}\right)a_0 = 0, & \text{if } c = 2e^{-\xi_0}\text{, the degenerate scale occurs} \\ a_1 = \frac{Sce^{\xi_0}}{2\pi\mu}, & n=1 \\ a_n = 0, & n=2,3,\ldots \\ b_n = 0, & n=1,2,3,4,\ldots \end{cases}$	$\begin{cases} a_0 = 0,\ C = 0 \\ a_1 = \frac{Sce^{\xi_0}}{2\pi\mu}, & n=1 \\ a_n = 0, & n=2,3,\ldots \\ b_n = 0, & n=1,2,3,4,\ldots \end{cases}$
The field solution of well-posed BVP	$u^M(\xi_x,\eta_x) = 2\pi\left(\xi_x + \ln\frac{c}{2}\right)a_0 + \frac{Sc}{\mu}e^{\xi_0-\xi_x}\cosh\xi_0\cos\eta_x$	$u^M(\xi_x,\eta_x) = \frac{Sc}{\mu}e^{\xi_0-\xi_x}\cosh\xi_0\cos\eta_x$
Total displacement	$u(\xi_x,\eta_x) = 2\pi\left(\xi_x + \ln\frac{c}{2}\right)a_0 + \frac{Sc}{\mu}\left(\cosh\xi_x + e^{\xi_0-\xi_x}\cosh\xi_0\right)\cos\eta_x$	$u(\xi_x,\eta_x) = \frac{Sc}{\mu}\left(\cosh\xi_x + e^{\xi_0-\xi_x}\cosh\xi_0\right)\cos\eta_x$

Conclusions

The solvability of solutions for the anti-plane shear problem, including a crack or a rigid-line inclusion, was investigated by using the direct and indirect BIEM/BEMs. The role of single-layer and double-layer approaches for solving double-degenerate BVPs was thoroughly examined. A rigid-line inclusion under the anti-plane shear $\sigma_{xz}^{\infty} = S$ can be solved by using the single-layer potential approach only, and a crack under the anti-plane shear $\sigma_{yz}^{\infty} = S$ can be solved by using the double-layer potential approach only. It is interesting to find that even though the influence matrix is rank-deficient due to the degenerate boundary, but we still can use only the single-layer or double-layer approach to solve the corresponding problem. Also, it is worth noting that even the unknown boundary data does not agree with the analytical solution, the solution of displacement field may still be obtained. We can explain it by using the SVD technique, the spectrum of boundary density can be decomposed into two parts, one is symmetric part and the other is anti-symmetric part. And according to the singular value of the influence matrix, one of the parts with a corresponding zero singular value will have no impact on the field solution. However, the problem of double degeneracy for symmetric cases cannot be solved by using the single-layer potential approach alone because of the degenerate scale. To overcome this problem, the Fichera formulation can be employed. Finally, the numerical results were compared with the exact solution to see the validity of our approach.

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