

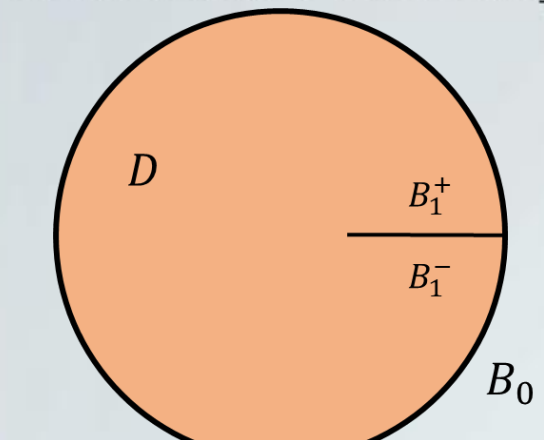


## 無網格邊界積分方程法於含裂縫扭轉桿的扭轉剛度分析

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**Abstract** 近年來由TKU/MSV所發展的無網格邊界積分方程法，經由局部正確解的引入，可以技巧性的計算柯西主值奇異積分。並透過參數化表示整個邊界路徑曲線，再使用高斯積分計算整個邊界積分，則邊界積分方程式可以直接地轉換成代數方程式。因此本文將使用無網格邊界積分方程法延伸處理含有裂縫的扭轉桿問題，當邊界配置點 $\mathbf{x}$ 與源點 $\mathbf{s}$ 相同( $r = |\mathbf{x} - \mathbf{s}| = 0$ )且位於裂縫時，會遭遇裂縫兩側同時發生奇異積分的狀況，因此如何找出適用於含有裂縫即退化邊界的局部正確解是本文的一大重點。為此我們通過引入橢圓座標系統來推導新的局部正確解，使其在裂縫兩側能夠技巧性的計算柯西主值奇異積分。本文考慮含裂縫的圓形斷面扭轉桿來驗證本法的正確性，並且考慮三種不同的裂縫形式，水平內裂縫、水平邊界裂縫、斜邊界裂縫，來比較其斷面扭轉剛度弱化的情況，最後發現水平邊界裂縫的情況弱化最多。

## Problem description



Stress function  
控制方程式  
 $\nabla^2 \Phi(\mathbf{x}) = -2G\alpha, \mathbf{x} \in D$   
邊界條件  
 $\Phi(\mathbf{x}) = \Phi_h(\mathbf{x}) + \Phi_p(\mathbf{x}) = 0, \mathbf{x} \in B$

Warping function  
控制方程式  
 $\nabla^2 w_a(\mathbf{x}) = 0, \mathbf{x} \in D$   
邊界條件  
 $\frac{\partial w_a(\mathbf{x})}{\partial x} n_x + \frac{\partial w_a(\mathbf{x})}{\partial y} n_y = \frac{\partial w_a(\mathbf{x})}{\partial n} = \alpha(yn_x - xn_y), \mathbf{x} \in B$

$G$ 是剪力模數  
 $\Phi(\mathbf{x})$ 為應力函數  
 $\Phi_h(\mathbf{x})$ 為補解  
 $\Phi_p(\mathbf{x})$ 為特解  
 $\nabla^2$ 是拉普拉斯運算子  
 $\alpha$ 是沿著 $z$ 軸的每單位長度的扭轉角

$w_a(\mathbf{x})$ 為翹曲函數  
 $D$ 為桿件斷面的領域  
 $(n_x, n_y)$ 為邊界 $B$ 上的單位向外法向量

Meshfree boundary integral equation method

$$0 = \int_B T(\mathbf{s}, \mathbf{x})[u(\mathbf{s}) - w(\mathbf{s})]dB(\mathbf{s}) - \int_B U(\mathbf{s}, \mathbf{x})[t(\mathbf{s}) - w_p(\mathbf{s})]dB(\mathbf{s}), \mathbf{x} \in B$$

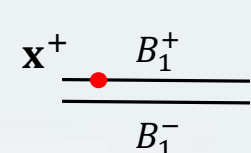
$$0 = \int_B M(\mathbf{s}, \mathbf{x})[u(\mathbf{s}) - w(\mathbf{s})]dB(\mathbf{s}) - \int_B L(\mathbf{s}, \mathbf{x})[t(\mathbf{s}) - w_p(\mathbf{s})]dB(\mathbf{s}), \mathbf{x} \in B$$

$$t(\mathbf{s}) = \frac{\partial u(\mathbf{s})}{\partial n(\mathbf{s})} \quad w_p(\mathbf{s}) = \frac{\partial w(\mathbf{s})}{\partial n(\mathbf{s})} \quad B = B_0 \cup B_1 \quad B_1 = B_1^+ \cup B_1^-$$

For  $\mathbf{x} \in B_0 \Rightarrow$  Ordinary boundary

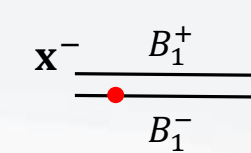
$$w(\mathbf{s}) = u(\mathbf{x}) + [(s_x - x)n_x(\mathbf{x}) + (s_y - y)n_y(\mathbf{x})]t(\mathbf{x})$$
$$0 = \sum_{j=1}^{N_B} \omega_j T(\mathbf{s}_j, \mathbf{x}_i)[u_j - w_j^i]j_b(\xi_j) - \sum_{j=1}^{N_B} \omega_j U(\mathbf{s}_j, \mathbf{x}_i)[t_j - w_p^i]j_b(\xi_j)$$

$$j_b(\xi) = \left( \frac{\tau_L - \tau_0}{2} \right) \sqrt{(x'(\tau))^2 + (y'(\tau))^2} \Big|_{\tau=\tau(\xi)}$$

For  $\mathbf{x} \in B_1 \Rightarrow$  Degenerate boundary

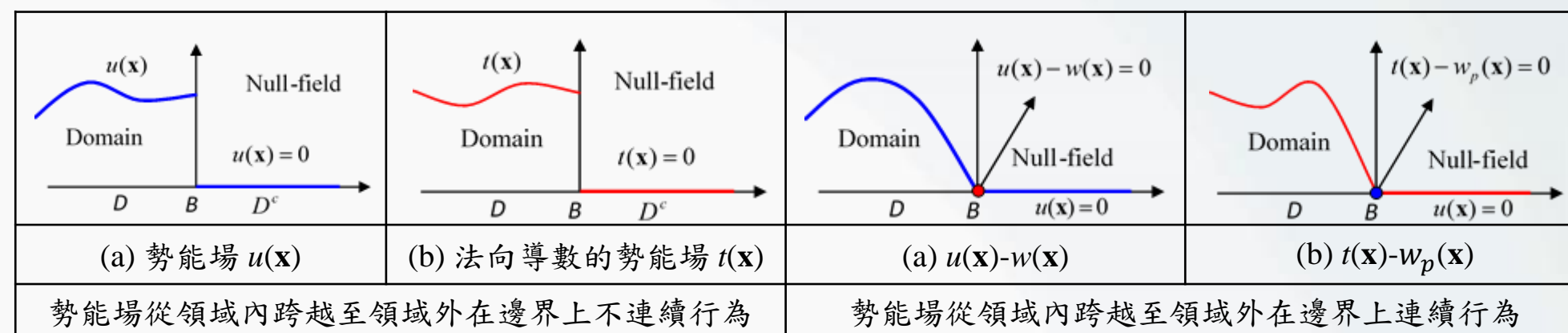
$$0 = \int_{B_1} T(\mathbf{s}, \mathbf{x})[u(\mathbf{s}) - w(\mathbf{s})]dB(\mathbf{s}) - \int_{B_1} U(\mathbf{s}, \mathbf{x})[t(\mathbf{s}) - w_p(\mathbf{s})]dB(\mathbf{s})$$
$$+ \int_{B_0} T(\mathbf{s}, \mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_{B_0} U(\mathbf{s}, \mathbf{x})t(\mathbf{s})dB(\mathbf{s}), \mathbf{x} \in B_1^+$$

$$w(\mathbf{s}) = f_1(\mathbf{s}, \mathbf{x}^+)u(\mathbf{x}^+) + f_2(\mathbf{s}, \mathbf{x}^+)u(\mathbf{x}^-) + g_1(\mathbf{s}, \mathbf{x}^+)t(\mathbf{x}^+) + g_2(\mathbf{s}, \mathbf{x}^+)t(\mathbf{x}^-)$$

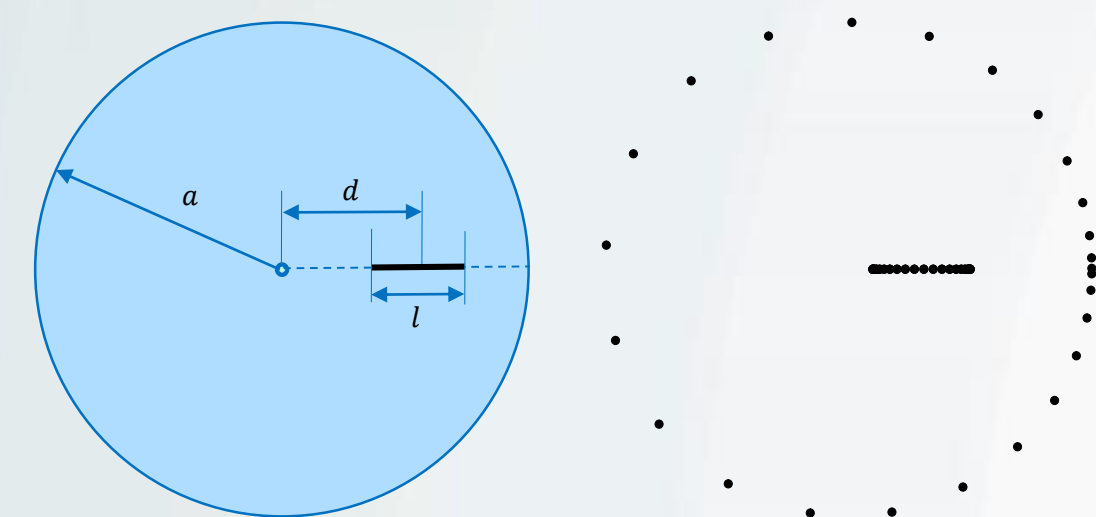


$$0 = \int_{B_1} M(\mathbf{s}, \mathbf{x})[u(\mathbf{s}) - w(\mathbf{s})]dB(\mathbf{s}) - \int_{B_1} L(\mathbf{s}, \mathbf{x})[t(\mathbf{s}) - w_p(\mathbf{s})]dB(\mathbf{s})$$
$$+ \int_{B_0} M(\mathbf{s}, \mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_{B_0} L(\mathbf{s}, \mathbf{x})t(\mathbf{s})dB(\mathbf{s}), \mathbf{x} \in B_1^-$$

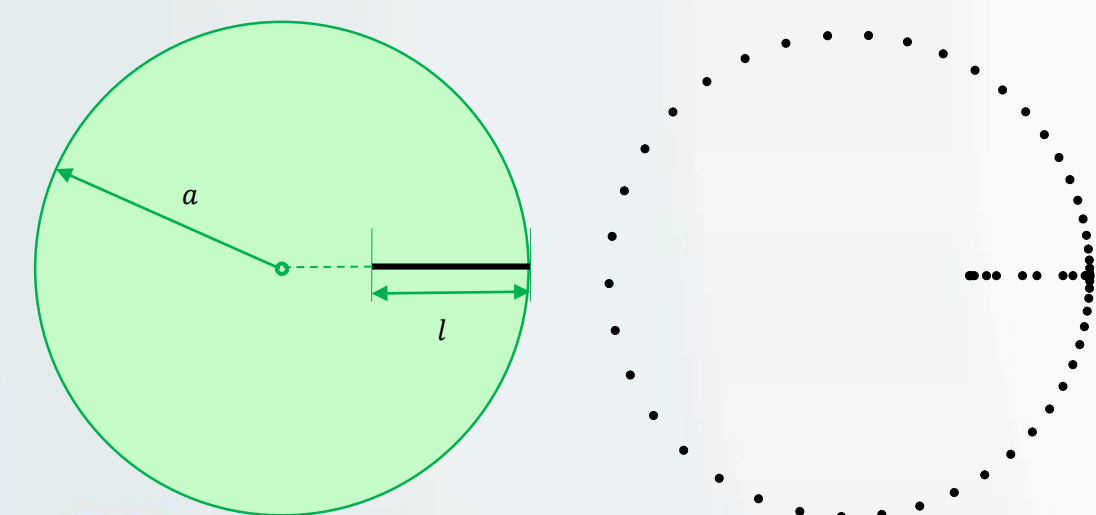
$$w(\mathbf{s}) = f_1(\mathbf{s}, \mathbf{x}^-)u(\mathbf{x}^+) + f_2(\mathbf{s}, \mathbf{x}^-)u(\mathbf{x}^-) + g_1(\mathbf{s}, \mathbf{x}^-)t(\mathbf{x}^+) + g_2(\mathbf{s}, \mathbf{x}^-)t(\mathbf{x}^-)$$



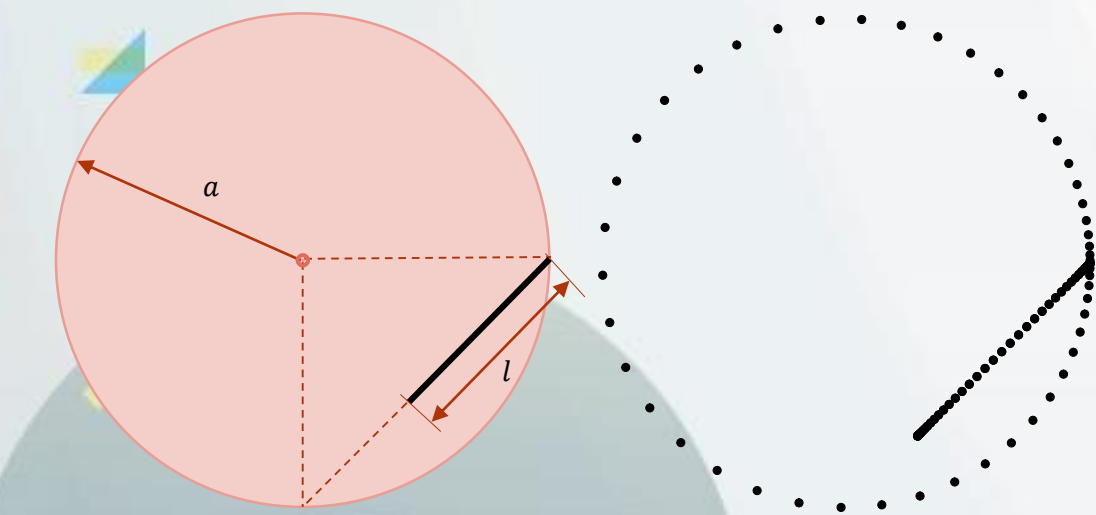
## Results and discussion



含水平內裂縫圓形斷面示意圖



含水平邊界裂縫圓形斷面示意圖



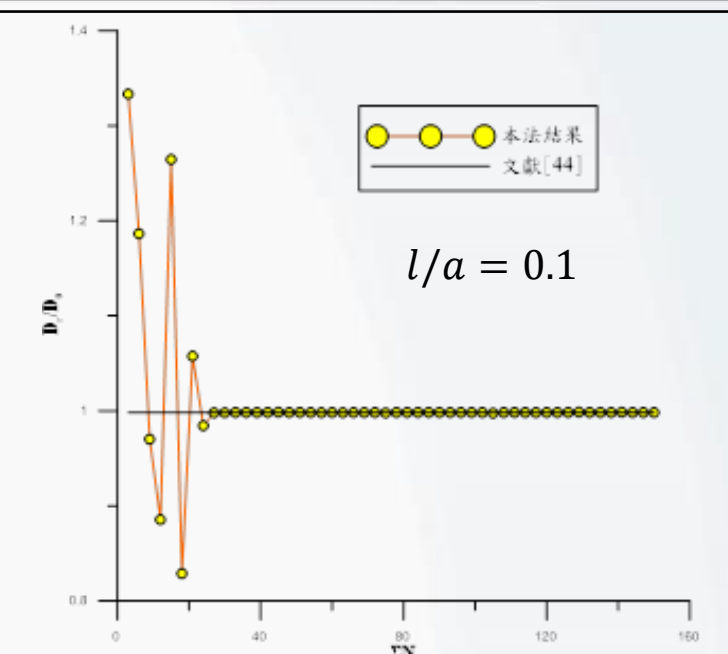
含斜向邊界裂縫圓形斷面示意圖

含水平內裂縫的圓形斷面扭轉桿其扭轉剛度比值 $D/D_0$ , $D_0 = G\frac{\pi}{2}a^4$					
$l/a$	本法 ( $\Sigma N = 900$ 點)	文獻[1]	文獻[2]	相對誤差(%)	
				和文獻[1]相比	和文獻[2]相比
0.1	0.9982	0.9981	0.9982	0.0064 %	0.0036 %
0.2	0.9922	0.9922	0.9922	0.0014 %	0.0014 %
0.3	0.9808	0.9808	0.9808	0.0044 %	0.0044 %
0.4	0.9612	0.9612	0.9612	0.0048 %	0.0048 %

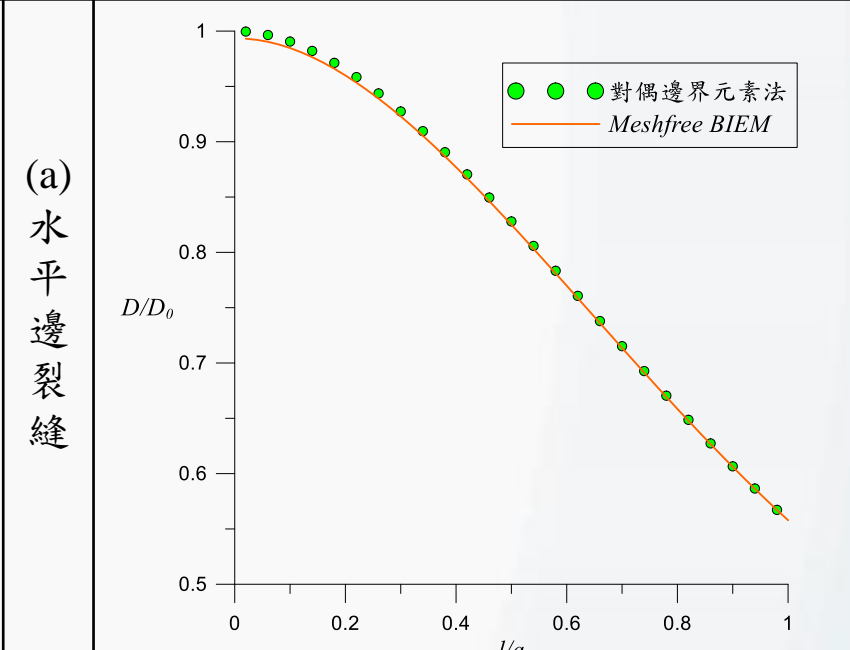
含水平邊界裂縫的圓形斷面扭轉桿其扭轉剛度比值 $D/D_0$ , $D_0 = G\frac{\pi}{2}a^4$				
$l/a$	本法 ( $\Sigma N = 150$ 點)	文獻[2]	文獻[3]	對偶邊界元素法 (總共150個元素)
1	0.5579	NA.	0.5590	0.5579
0.9	0.6066	0.6076	NA.	0.6061
0.8	0.6595	0.6603		0.6585
0.7	0.7152	0.7159		0.7136
0.6	0.7720	0.7727		0.7698
0.5	0.8279	0.8285	0.8149	0.8250

含斜向邊界裂縫的圓形斷面扭轉桿其扭轉剛度比值 $D/D_0$ , $D_0 = G\frac{\pi}{2}a^4$		
$l/a$	本法 ( $\Sigma N = 150$ 點)	對偶邊界元素法 (總共150個元素)
1	0.8178	0.8145
0.9	0.8305	0.8271
0.8	0.8470	0.8435
0.7	0.8669	0.8631
0.6	0.8894	0.8854
0.5	0.9137	0.9093

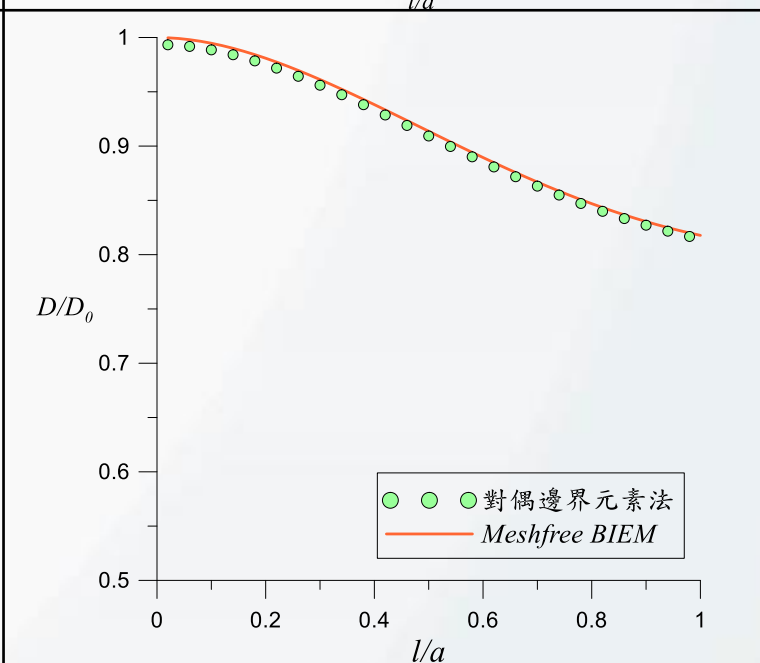
本法與文獻比較的收斂圖



扭轉剛度比值與無因次化裂縫長度的關係圖



(b) 斜向邊界裂縫



## Conclusions

1. 透過參數化表示曲線，並採用高斯積分轉換邊界積分方程式，即可無需建構元素網格。
2. 通過引入橢圓座標系統，可以找出適用於含有裂縫即退化邊界的局部正確解。
3. 本法所得出數值結果，其相對誤差與參考文獻相比後有良好的收斂性。
4. 含水平裂縫的情況，對其扭轉剛度的影響較大。

$$f_1(\mathbf{s}, \mathbf{x}^+) = \frac{e^{\xi x}}{\cos(\eta_{x^+})} e^{-\xi s} \cos(\eta_s) - \frac{1}{2} \frac{e^{2\xi x}}{\cos(2\eta_{x^+})} e^{-2\xi s} \cos(2\eta_s)$$
$$+ \frac{e^{\xi x}}{\sin(\eta_{x^+})} e^{-\xi s} \sin(\eta_s) - \frac{1}{2} \frac{e^{2\xi x}}{\sin(2\eta_{x^+})} e^{-2\xi s} \sin(2\eta_s)$$
$$f_2(\mathbf{s}, \mathbf{x}^+) = \frac{e^{\xi x}}{\cos(\eta_{x^+})} e^{-\xi s} \cos(\eta_s) - \frac{1}{2} \frac{e^{2\xi x}}{\cos(2\eta_{x^+})} e^{-2\xi s} \cos(2\eta_s)$$
$$- \frac{e^{\xi x}}{\sin(\eta_{x^+})} e^{-\xi s} \sin(\eta_s) + \frac{1}{2} \frac{e^{2\xi x}}{\sin(2\eta_{x^+})} e^{-2\xi s} \sin(2\eta_s)$$
$$g_1(\mathbf{s}, \mathbf{x}^+) = -\frac{1}{2} J(\mathbf{x}^+) \left[ \frac{e^{\xi x}}{\cos(\eta_{x^+})} e^{-\xi s} \cos(\eta_s) - \frac{e^{2\xi x}}{\cos(2\eta_{x^+})} e^{-2\xi s} \cos(2\eta_s) \right]$$
$$+ \frac{e^{\xi x}}{\sin(\eta_{x^+})} e^{-\xi s} \sin(\eta_s) - \frac{e^{2\xi x}}{\sin(2\eta_{x^+})} e^{-2\xi s} \sin(2\eta_s)$$
$$g_2(\mathbf{s}, \mathbf{x}^+) = -\frac{1}{2} J(\mathbf{x}^+) \left[ \frac{e^{\xi x}}{\cos(\eta_{x^+})} e^{-\xi s} \cos(\eta_s) - \frac{e^{2\xi x}}{\cos(2\eta_{x^+})} e^{-2\xi s} \cos(2\eta_s) \right]$$
$$- \frac{e^{\xi x}}{\sin(\eta_{x^+})} e^{-\xi s} \sin(\eta_s) + \frac{e^{2\xi x}}{\sin(2\eta_{x^+})} e^{-2\xi s} \sin(2\eta_s)$$

## References

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