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An analytical Green's function for an infinite plane with two circular holes using degenerate kernels

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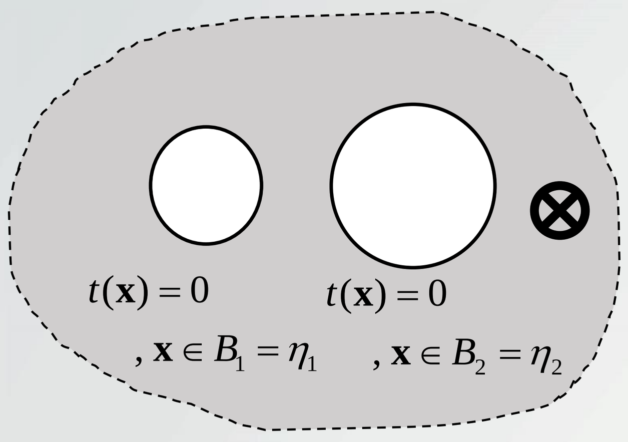
Abstract

In this poster, Green's function problem containing infinite plane with two circular Neumann boundaries is analytically studied by using boundary integral equation method (BIEM). The problem is decomposed into two parts. The first part is a free field caused by the concentric force. The second part is a boundary value problem subjected to corresponding boundary conditions. The second part can be solved by using the null-filed boundary integral equation in conjunction with the degenerate kernel. Since the geometry of interested problem is containing two circular boundaries, the kernel function is expanded to series form in terms of the bipolar coordinates. Once the field of boundary value problem is solved, the total field can be obtained by superimposing the free field and the field of boundary value problem. To show the validity of the present method, the contour result of the present method is compared with those done by the image method, null-field BIE in conjunction with adaptive observer.

Problem description

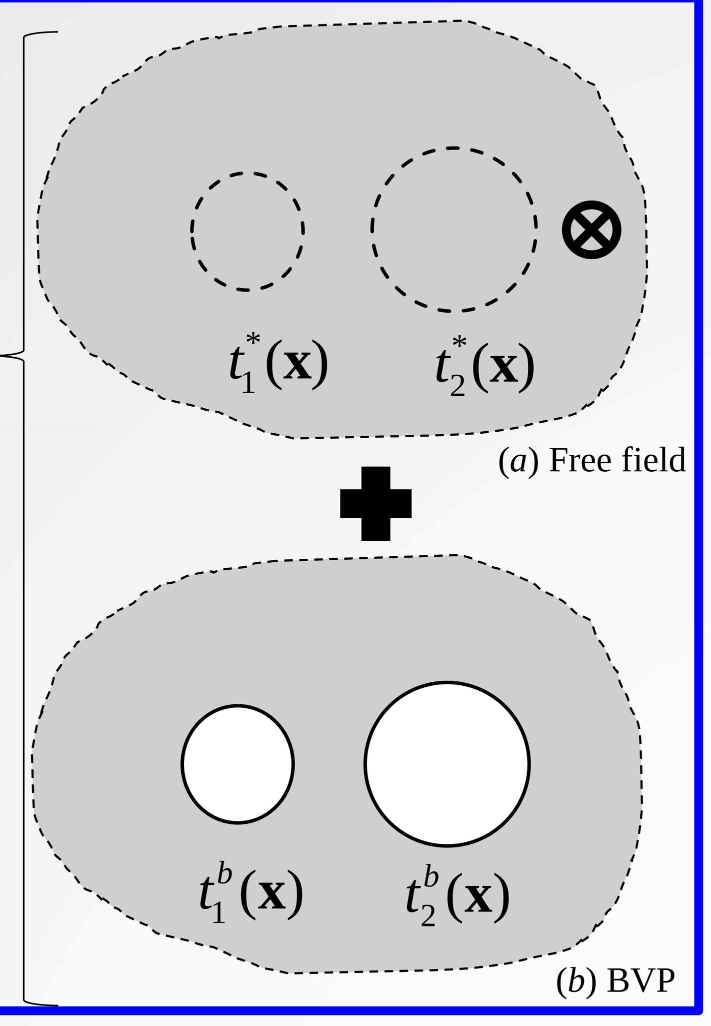


Governing equation:

$$\nabla^2 w(\mathbf{x})=\delta(\mathbf{x}-\mathbf{s}^*),\ \mathbf{x}\in D$$

Boundary condition:

$$t(\mathbf{x})=\frac{\partial w(\mathbf{x})}{\partial n_{\mathbf{x}}}=0,\ \mathbf{x}\in B_i, i=1,2$$



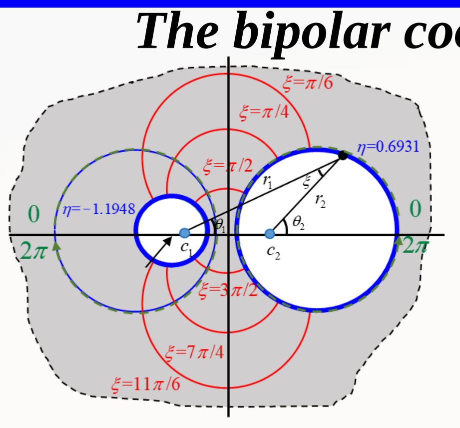
Boundary integral equation

$$2\pi w_B(\mathbf{x})=\int_B T_{dk}(\mathbf{s},\mathbf{x})w_B(\mathbf{s})dB(\mathbf{s})-\int_B U_{dk}(\mathbf{s},\mathbf{x})t_B(\mathbf{s})dB(\mathbf{s}),\ \mathbf{x}\in D\cup B$$

Null-field integral equation

$$0=\int_B T_{dk}(\mathbf{s},\mathbf{x})w_B(\mathbf{s})dB(\mathbf{s})-\int_B U_{dk}(\mathbf{s},\mathbf{x})t_B(\mathbf{s})dB(\mathbf{s}),\ \mathbf{x}\in D^c\cup B$$

The bipolar coordinates



$$x=c\frac{\sinh(\eta)}{\cosh(\eta)-\cos(\xi)}$$

$$y=c\frac{\sin(\eta)}{\cosh(\eta)-\cos(\xi)}$$

$$\eta=\ln r_1-\ln r_2$$

$$\xi=\theta_2-\theta_1$$

$\xi,\ 0\le\xi\le 2\pi$

$\eta,\ -\infty<\eta<\infty$

Degenerate kernel

$$U_{dk}(\mathbf{s},\mathbf{x})=\ln|\mathbf{s}-\mathbf{x}|=\begin{cases}\ln(2c)-\eta_s-\sum_{m=1}^{\infty}\frac{1}{m}[e^{-m(\eta_x-\eta_s)}\cos m(\xi_x-\xi_s)-e^{-m\eta_x}\cos m\xi_x-e^{-m\eta_s}\cos m\xi_s], & \eta_x\ge\eta_s\ge 0, \quad (a)\\ \ln(2c)-\eta_x-\sum_{m=1}^{\infty}\frac{1}{m}[e^{-m(\eta_s-\eta_x)}\cos m(\xi_x-\xi_s)-e^{-m\eta_x}\cos m\xi_x-e^{-m\eta_s}\cos m\xi_s], & \eta_s>\eta_x>0, \quad (b)\\ \ln(2c)-\sum_{m=1}^{\infty}\frac{1}{m}[e^{-m(\eta_s-\eta_x)}\cos m(\xi_x-\xi_s)-e^{m\eta_x}\cos m\xi_x-e^{-m\eta_s}\cos m\xi_s], & \eta_s>0>\eta_x, \quad (c)\\ \ln(2c)+\eta_s-\sum_{m=1}^{\infty}\frac{1}{m}[e^{-m(\eta_s-\eta_x)}\cos m(\xi_x-\xi_s)-e^{m\eta_x}\cos m\xi_x-e^{m\eta_s}\cos m\xi_s], & \eta_x\le\eta_s<0, \quad (d)\\ \ln(2c)+\eta_x-\sum_{m=1}^{\infty}\frac{1}{m}[e^{-m(\eta_x-\eta_s)}\cos m(\xi_x-\xi_s)-e^{m\eta_x}\cos m\xi_x-e^{m\eta_s}\cos m\xi_s], & \eta_s<\eta_x<0, \quad (e)\\ \ln(2c)-\sum_{m=1}^{\infty}\frac{1}{m}[e^{-m(\eta_x-\eta_s)}\cos m(\xi_x-\xi_s)-e^{-m\eta_x}\cos m\xi_x-e^{m\eta_s}\cos m\xi_s], & \eta_s<0<\eta_x \quad (f)\end{cases}$$

$$T_{dk}(\mathbf{s},\mathbf{x})=\frac{\partial U_{dk}(\mathbf{s},\mathbf{x})}{\partial n_x}$$

Generalized Fourier series for boundary densities

$$w_b(\mathbf{s})=a_0+\sum_{n=1}^{\infty}a_n\cos n\eta_s+\sum_{n=1}^{\infty}b_n\sin n\eta_s,\ 0\le\eta_s<2\pi,\ \mathbf{s}=(\xi_s,\eta_s)\in B$$

$$t_b(\mathbf{s})=\frac{1}{J_s}\left(p_0+\sum_{n=1}^{\infty}p_n\cos n\eta_s+\sum_{n=1}^{\infty}q_n\sin n\eta_s\right),\ 0\le\eta_s<2\pi,\ \mathbf{s}=(\xi_s,\eta_s)\in B$$

Results and discussion

Analytical solution

For $\eta_2\ge\eta_x>\eta_*>0$

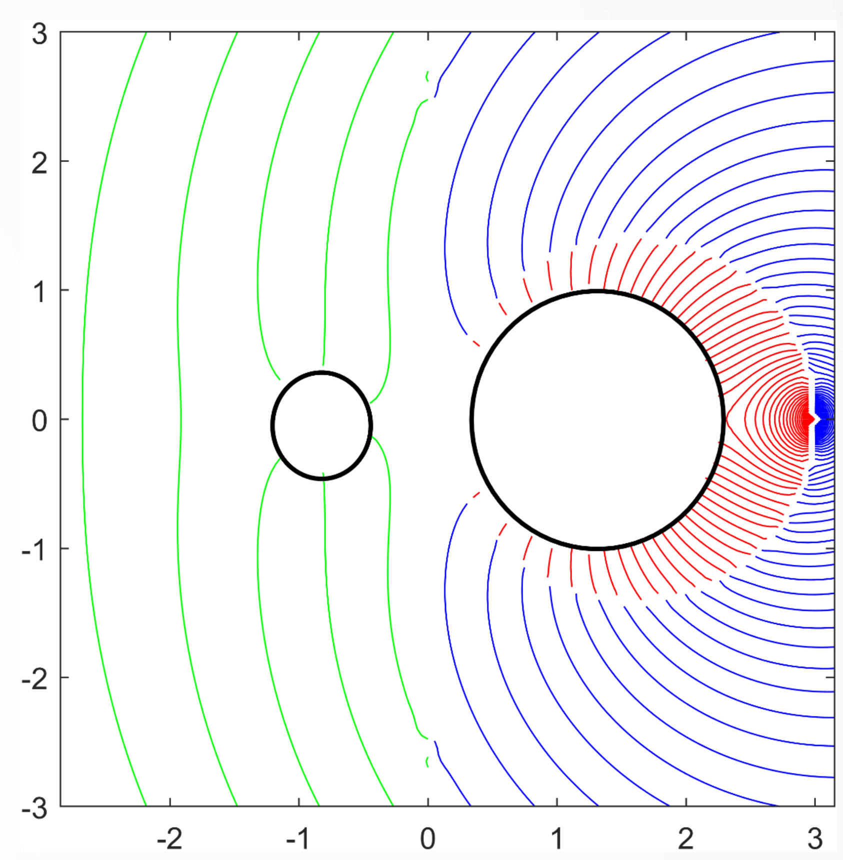
$$w(\mathbf{x})=\ln(2c)-\eta_*+\sum_{n=1}^{\infty}\frac{1}{n}\Big[\frac{-e^{-n(\eta_x-\eta_*)}-e^{-n(\eta_x+\eta_*+2\eta_1)}-e^{-n(-\eta_x-\eta_*+2\eta_2)}-e^{-n(-\eta_x+\eta_*+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n(\xi_x-\xi_*)+\frac{e^{-n\eta_x}+e^{-n(\eta_x+2\eta_1)}+e^{-n(-\eta_x+2\eta_2)}+e^{-n(-\eta_x+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n\xi_x$$
$$+\frac{e^{-n\eta_*}+e^{-n(\eta_*+2\eta_1)}+e^{-n(-\eta_*+2\eta_2)}+e^{-n(-\eta_*+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n\xi_*+\frac{-e^{-2n\eta_1}-e^{-2n\eta_2}-2e^{-n(2\eta_1+2\eta_2)}}{n(1-e^{-2n(\eta_1+\eta_2)})}\Big]$$

For $\eta_*\ge\eta_x>0$

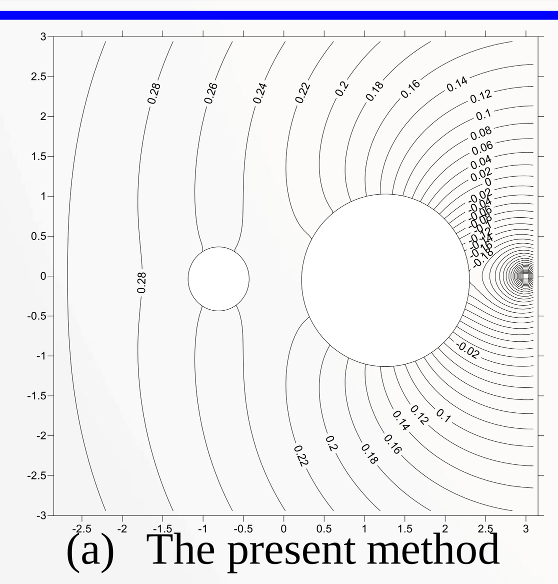
$$w(\mathbf{x})=\ln(2c)-\eta_*+\sum_{n=1}^{\infty}\frac{1}{n}\Big[\frac{-e^{-n(\eta_*-\eta_x)}-e^{-n(\eta_x+\eta_*+2\eta_1)}-e^{-n(-\eta_x-\eta_*+2\eta_2)}-e^{-n(\eta_x-\eta_*+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n(\xi_x-\xi_*)+\frac{e^{-n\eta_x}+e^{-n(\eta_x+2\eta_1)}+e^{-n(-\eta_x+2\eta_2)}+e^{-n(-\eta_x+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n\xi_x$$
$$+\frac{e^{-n\eta_*}+e^{-n(\eta_*+2\eta_1)}+e^{-n(-\eta_*+2\eta_2)}+e^{-n(-\eta_*+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n\xi_*+\frac{-e^{-2n\eta_1}-e^{-2n\eta_2}-2e^{-n(2\eta_1+2\eta_2)}}{n(1-e^{-2n(\eta_1+\eta_2)})}\Big]$$

For $\eta_*>0\ge\eta_x\ge\eta_1$

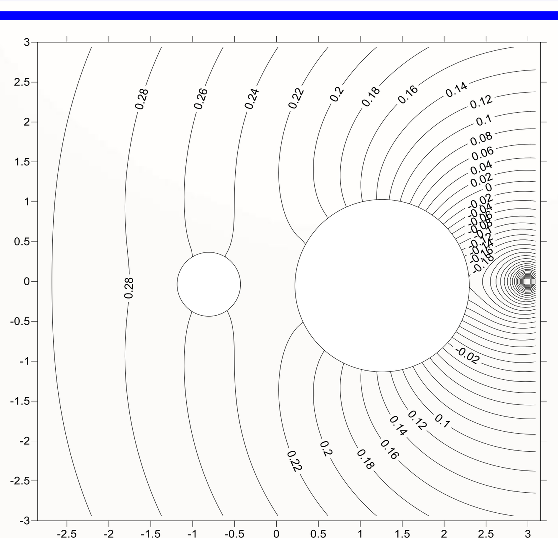
$$w(\mathbf{x})=\ln(2c)-\eta_*+\sum_{n=1}^{\infty}\frac{1}{n}\Big[\frac{-e^{-n(\eta_*-\eta_x)}-e^{-n(\eta_x+\eta_*+2\eta_1)}-e^{-n(-\eta_x-\eta_*+2\eta_2)}-e^{-n(\eta_x-\eta_*+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n(\xi_x-\xi_*)+\frac{e^{n\eta_x}+e^{-n(\eta_x+2\eta_1)}+e^{-n(-\eta_x+2\eta_2)}+e^{-n(\eta_x+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos n\xi_x$$
$$+\frac{e^{-n\eta_*}+e^{-n(\eta_*+2\eta_1)}+e^{-n(-\eta_*+2\eta_2)}+e^{-n(-\eta_*+2\eta_1+2\eta_2)}}{(1-e^{-2n(\eta_1+\eta_2)})}\cos(n\xi_*)+\frac{-e^{-2n\eta_1}-e^{-2n\eta_2}-2e^{-n(2\eta_1+2\eta_2)}}{n(1-e^{-2n(\eta_1+\eta_2)})}\Big]$$



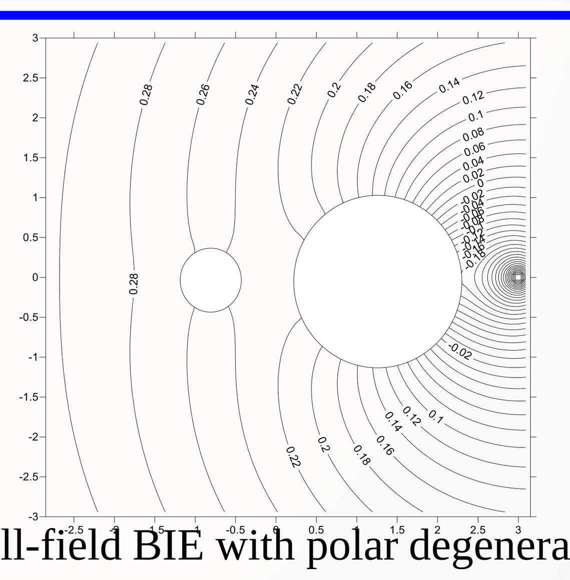
Contour plot comparison



(a) The present method
(Null-field BIE with degenerate kernel of the bipolar coordinates)



(b) The image method [1]



(c) Null-field BIE with polar degenerate kernel and adaptive observer [2]

Conclusions

The infinite plane with two Neumann circular boundaries is solved by the present method. The kernel function is expanded to series form instead of closed-form fundamental solution. Since the problem is containing with two circular boundaries, the degenerate kernel is expanded under the bipolar coordinates. By using the present methodology, the analytical solution of the problem is derived. To show the validity of the present method, the contour plot of present formulation is plot and compared with those done by the image method and null-field BIE with polar degenerate kernel. Good agreement is made.

References

[1] Chen JT, Shieh HC, Lee YT, Lee JW. Bipolar coordinates, image method and the method of fundamental solutions for Green's functions of Laplace problems containing circular boundaries. Eng. Anal. Bound. Elem. 35:236–243, 2011.

[2] Chen JT, Chou KH, Kao SK. Derivation of Green's function using addition theorem. Mech. Res. Commun., 36: 351–363, 2009.