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Active Set Method for Solving Convex Quadratic Programs

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1. Abstract

Quadratic Programming (QP) can be applied in many fields such as finance, agriculture, economy, etc. In our research, we will propose a general-purpose solver for convex quadratic programs with linear inequality constraints based on the active set method, and present the numerical results compared with other existing solver.

2. Introduction

We want to solve the quadratic program in the form

$$\min_{Ax\le b} f(x) = \frac{1}{2}x^T Hx + c^T x, \tag{1}$$

where $x \in \mathbb{R}^n$. The objective function f is given by $H \in S_+^n$ and $c \in \mathbb{R}^n$, the feasible set denoted by $X = \{x \mid Ax \le b\}$ is given by $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

Since $H \in S_+^n$, H is orthogonal equivalent to a nonnegative real diagonal matrix. In order to simplify the notation, we denote $H = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \in S_+^n$, where $D \in S_{++}^r$ is real diagonal, $r = rank(H)$; $A = [B|N]$ with $B \in \mathbb{R}^{m\times r}$ and $N \in \mathbb{R}^{m\times(n-r)}$; $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ with $c_1 \in \mathbb{R}^r$ and $c_2 \in \mathbb{R}^{(n-r)}$; $x = \begin{bmatrix} z \\ w \end{bmatrix}$ with $z \in \mathbb{R}^r$ and $w \in \mathbb{R}^{(n-r)}$.

3. Idea

We consider the dual problem of (1) be the problem (2), regularized problem (1) be the problem (3), and consider the dual problem of (3) be the problem (4). Then can solve problem (4) instead of problem (1) by active set method and implicitly take ϵ be zero.

$$\begin{tikzcd}
(1)\min_{Ax\le b} \frac{1}{2}x^T Hx + x^T c \arrow[r, leftrightarrow, "\text{Strong Duality}"] & (2)\max_{\substack{N^T u = c_2 \\ u\ge 0}} -\frac{1}{2}\|B^T u + c_1\|^2_{(D^{-1})} - u^T b \\
(3)\min_{Ax\le b} \frac{1}{2}x^T(H+\epsilon I)x + x^T c \arrow[u, "\text{Implicitly let } \epsilon\to 0"] \arrow[r, leftrightarrow, "\text{Strong Duality}"] & \max_{u\ge 0} -\frac{1}{2}\|B^T u + c_1\|^2_{(D^{-1})} - \frac{1}{2\epsilon}\|N^T u - c_2\|^2 - u^T b \arrow[u, "\text{Implicitly let } \epsilon\to 0"']
\end{tikzcd}$$

4. Examples

We randomly generated matrices in MATLAB, and compared with *OSQP* to check the optimal objective value '*Cost*' and iteration steps '*Iteration*'. We show two examples in the table, one is fix $n = 30$, $m = 30$, and change $r = 30, 15, 0$. Particularly, our problem will become a strictly convex quadratic programs as $r = 30$, and becomes a linear programs as $r = 0$. The other example is fix $n = 30$, $r = 15$, and change $m = 15, 30, 60$.

Table 1: Fix n, $m = 30$

	ISM		OSQP	
r	Cost	Iteration	Cost	Iteration
30	-1.60e+01	5	-1.60e+01	75
15	-1.16e+02	9	-1.16e+02	75
0	-1.69e+01	4	-1.69e+01	50

Table 2: Fix $n = 30$, $r = 15$

	ISM		OSQP	
m	Cost	Iteration	Cost	Iteration
15	-1.08e+02	6	-1.08e+02	75
30	-4.65e+01	6	-4.65e+01	100
60	1.66e+01	18	1.66e+01	150

5. Future Work

We would like to develop a more efficient algorithm to solve this problem in the future.

6. Reference

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