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Spreading speeds for a three-species competition-diffusion system with monotonicity

Author: Jyun-Sheng Jiang (江俊昇), Department of Applied Mathematics, National Yang Ming Chiao Tung University, Taiwan (a0987874923@gmail.com)
Advisor: Prof. Chang-Hong Wu (吳昌鴻)

Abstract

We investigates the propagation properties of a competition-diffusion system involving three species with monotonicity. The system describes the invasion of an open habitat by three competing species, in which there is no interaction between two of the species, and the competition between all species is intense. Herein, with the help of traveling waves, we will construct several appropriate comparison functions, utilize the concept of sub- and super-solutions, and apply the comparison principle to prove the survival and extinction of the species in different regions with different spreading speeds. Furthermore, we can characterize the spreading speeds of each species under appropriate conditions. Our findings indicate that species expansion can be intricate, involving at least four distinct propagation modes.

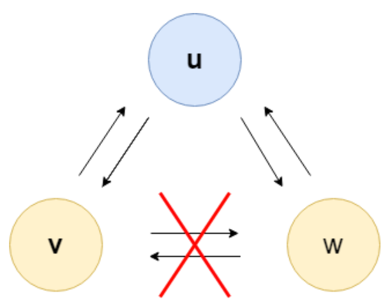
Problem description

Reaction-diffusion equations are fundamental mathematical models with broad applications in many fields, such as biology, chemistry, physics, etc. The Fisher-KPP equation is a well-known model that is commonly used to describe the spread or invasion of a species in open space [1,3]. For multiple species case, the spreading phenomenon will be more complicated, for example, [2] considered two-species competition model.

Here we consider a three-species system that generalizes the work [1,2,3]. To make this system amenable to the comparison principle, we consider the following simplified three-species competition-diffusion system between species u, v, and w:

$$\begin{cases} \partial_t u = \partial_{xx}^2 u + u\left(1 - u - c_{12}v - c_{13}w\right), & x \in \mathbb{R},\ t > 0, \\ \partial_t v = d_2 \partial_{xx}^2 v + r_2 v\left(1 - v - c_{21}u\right), & x \in \mathbb{R},\ t > 0, \\ \partial_t w = d_3 \partial_{xx}^2 w + r_3 w\left(1 - w - c_{31}u\right), & x \in \mathbb{R},\ t > 0, \end{cases} \qquad (1)$$

where we have assumed that there is no competition between the two species v and w.



Goal: understand the spreading behavior of the system with suitable initial conditions when the competition between species is strong enough. For this, we make the following assumptions:

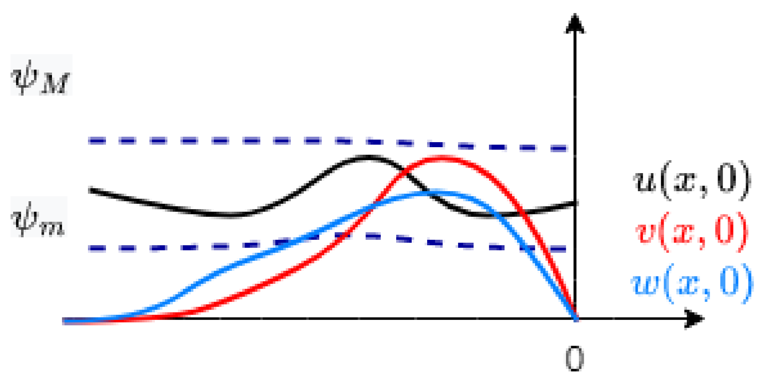
(A1) $c_{12} > 1, c_{13} > 1, c_{21} > 1$ ***and*** $c_{31} > 1$.

The condition **(A1)** indicates that the competition between every two species is strong.

To focus on the spreading behavior to the right, we consider the initial data:

$$\begin{cases} u_0(x) = v_0(x) = w_0(x) = 0 \text{ for } x \geq 0, \\ 0 < \psi_m \leq u_0(x) \leq \psi_M < 1 \text{ for } x < 0, \\ 0 \leq (\not\equiv) v_0(x) < 1 \text{ for } x < 0 \text{ and } v_0(x) = O\left(e^{\lambda x}\right) \text{ as } x \to -\infty, \\ 0 \leq (\not\equiv) w_0(x) < 1 \text{ for } x < 0 \text{ and } w_0(x) = O\left(e^{\lambda x}\right) \text{ as } x \to -\infty, \end{cases} \qquad (2)$$

where ψ_M and ψ_m are some positive constants and λ is given a sufficient large constant.



Key quantities:

$c_u := 2$, $c_v = 2\sqrt{r_2 d_2}$, $c_w = 2\sqrt{r_3 d_3}$

c_{uv}: the traveling wave speed of the system (1) with w ≡ 0 (cf. [5])

c_{uw}:the traveling wave speed of the system (1) with v ≡ 0

c_{uvw}: the traveling wave speed of the system (1)

In addition to the assumption **(A1),** to simplify our analysis, we further assume

(A2) $c_{uvw} > 0, c_{uv} > 0$ ***and*** $c_{uw} > 0$.

We can show that (A2) is not void.

Main results

We found that the spreading behavior of species strongly depends on

$$c_u := 2,\ c_v = 2\sqrt{r_2 d_2},\ c_w = 2\sqrt{r_3 d_3}$$

Due to the symmetry of the system (1), without loss of generality, we may assume that

$$c_v < c_w.$$

Theorem 1. *Assume that* **(A1)** *and* **(A2)** *hold. Let* (u, v, w) *be a solution of (1)-(2). Then the solution* (u, v, w) *satisfies*

$$\forall c > c_u,\ \lim_{t\to+\infty} \sup_{x>ct} |u(x,t)| = 0;\ \forall c > c_v,\ \lim_{t\to+\infty} \sup_{x>ct} |v(x,t)| = 0;\ \forall c > c_w,\ \lim_{t\to+\infty} \sup_{x>ct} |w(x,t)| = 0,$$

$$\forall 0 < c < c_{uvw},\quad \lim_{t\to+\infty} \sup_{x<ct} |u(x,t) - 1| + |v(x,t)| + |w(x,t)| = 0.$$

Moreover,

(a) *if* $c_v < c_w < c_u$, *then*

$$\forall c < c_u,\ \lim_{t\to+\infty} \sup_{x<ct} |u(x,t) - 1| + |v(x,t)| + |w(x,t)| = 0.$$

(b) *if* $c_u < c_v < c_w$, *then*

$$(1.1) \qquad \begin{cases} \forall c_{uvw} < c_1 < c_2 < c_v, & \lim\limits_{t\to+\infty} \sup\limits_{c_1 t < x < c_2 t} |u(x,t)| + |v(x,t) - 1| = 0, \\ \forall c_{uvw} < c_1 < c_2 < c_w, & \lim\limits_{t\to+\infty} \sup\limits_{c_1 t < x < c_2 t} |u(x,t)| + |w(x,t) - 1| = 0. \end{cases}$$

(c) *if* $c_v < c_u < c_w$, *then the following hold:*

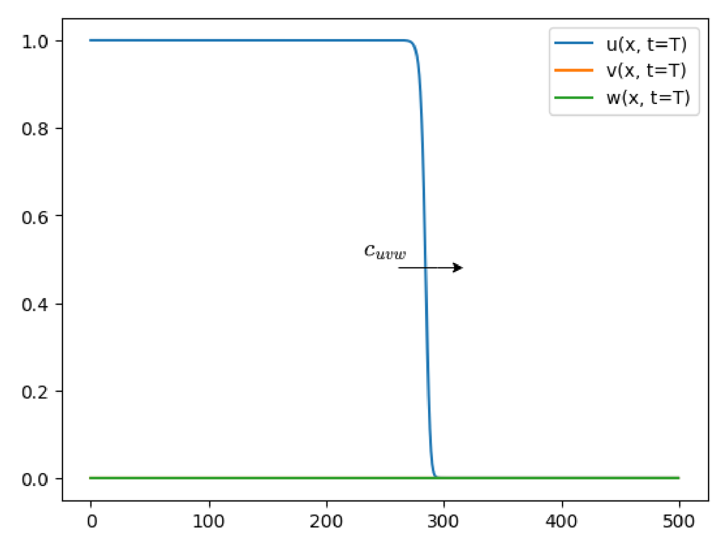
(i) *when* $c_{uw} < c_v$, *then* (1.1) *holds;*

(ii) *when* $c_{uvw} > c_v$, *then* $\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |v(x,t)| = 0$, *and*

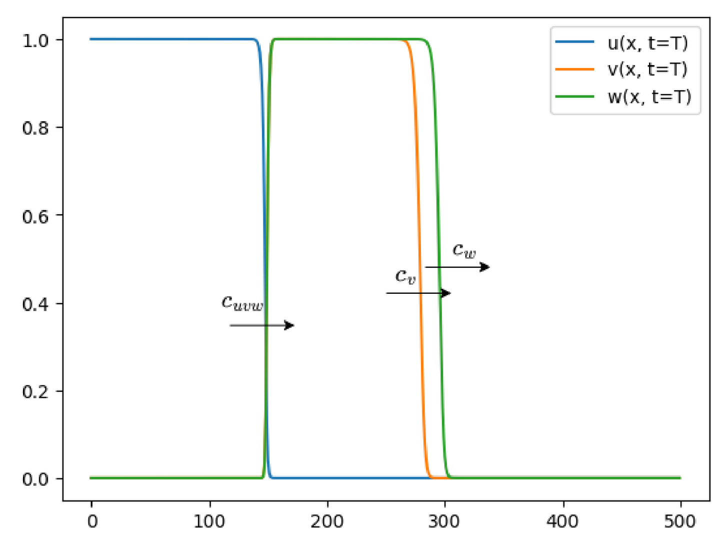
$$\forall c_{uw} < c_1 < c_2 < c_w,\ \lim_{t\to+\infty} \sup_{c_1 t < x < c_2 t} |u(x,t)| + |w(x,t) - 1| = 0,$$

$$\forall 0 < c < c_{uw},\ \lim_{t\to+\infty} \sup_{x<ct} |u(x,t) - 1| + |w(x,t)| = 0.$$

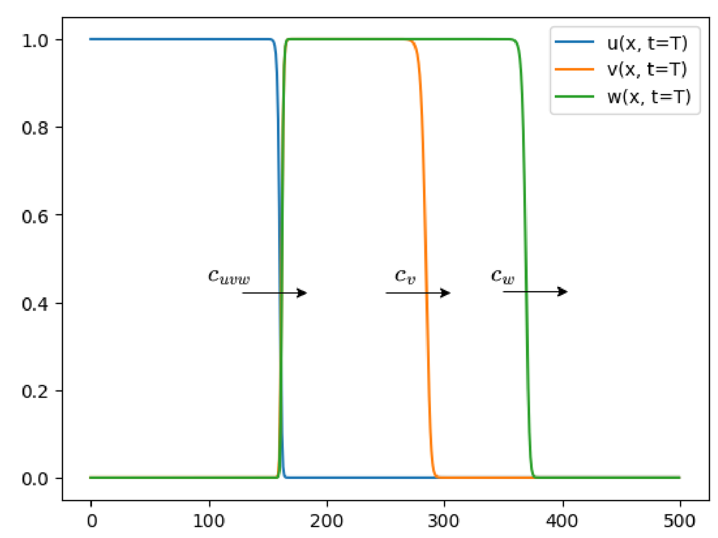
Numerical examples



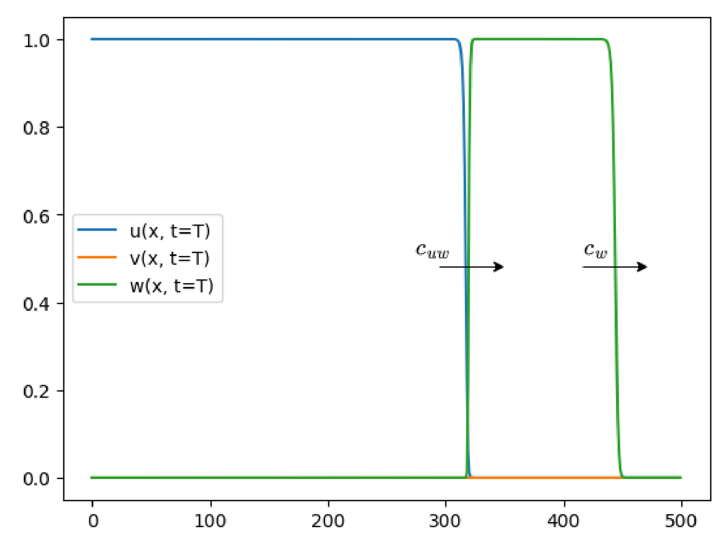
(i) If $c_u > \max\{c_v, c_w\}$



(ii) If $c_w > c_v > c_u$



(iii) If $c_w > c_u > c_v$ *and* $c_v > c_{uw}$



(iv) If $c_w > c_u > c_v$ *and* $c_{uvw} > c_v$

Conclusion

We have found that with different conditions, there are four different spreading behaviors appearing in the three-species competition model (1). More precisely, we have

(1) under the condition $c_v < c_w < c_u$, our result shows the species u spreads at a speed of c_u; both the species v and w spread at speed 0 and die out eventually.

(2) under the condition $c_u < c_v < c_w$, our result shows that the species u spreads at a speed of c_{uvw} ; the species v spreads at a speed of c_v; the species w spreads at a speed of c_w.

(3) under the condition $c_v < c_u < c_w$, our result shows that the species u can spread at a speed of c_{uvw} or c_{uw} ; the species v spreads at a speed of c_v or 0 ; the species w spreads at a speed of c_w .

References

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