Combinatorics Qualifying Examination

NTNU Math Ph.D. Program, Fall 2019

1. (10%) What is the expected number of fixed points of a permutation in $S(n)$?

2. (10%) Let a_n be the number of n-words over the alphabet $\{0, 1, 2\}$ that contain no neighboring 0's, e.g., $a_1 = 3$, $a_2 = 8$, $a_3 = 22$. Find the generating function of a_n.

3. (15%) Let a_n be the number of self-conjugate partitions of n. Prove the following identities:

   (a) $\displaystyle\sum_{n\geq 0} a_n z^n = \prod_{i\geq 1}(1 + z^{2i-1})$.

   (b) $\displaystyle\sum_{n\geq 0} \frac{q^n z^{n^2}}{(1-z^2)(1-z^4)\cdots(1-z^{2n})} = \prod_{i\geq 1}(1 + qz^{2i-1})$

   (c) $\displaystyle\prod_{i\geq 1}(1 + z^i) = \prod_{i\geq 1}(1 - z^{2i-1})^{-1}$

4. Let $i_n^{(r)}$ be the number of permutations in $S(n)$ with no cycles of length greater than r.

   (a) (5%) Prove $i_{n+1}^{(2)} = i_n^{(2)} + n i_{n-1}^{(2)}$.

   (b) (10%) Prove $i_{n+1}^{(r)} = \displaystyle\sum_{k=n-r+1}^{n} n^{\underline{n-k}} i_k^{(r)}$.

5. (10%) A permutation $\sigma \in S(n)$ is called connected if for any k, $1 \leq k < n$, $\{\sigma(1), \sigma(2), \dots, \sigma(k)\} \neq [k]$. Find the number of connected permutations in $S(8)$.

6. (10%) Toss a fair coin until you get heads for the n-th time. Let X be the number of throws necessary. What are $P_X(z)$, $E(X)$, and $Var(X)$?

7. (10%) Let a_n be the number of ordered set partitions of $\{1, \dots, n\}$. Compute $\displaystyle\sum_{n\geq 0} a_n \frac{z^n}{n!}$.

8. (10%) Let S be the family of k-subsets of $\{1, 2, \dots, 2n\}$. For $A \in S$ let $w(A) = \displaystyle\sum_{i\in A} i$, and set $S^+ = \{A \in S \mid w(A) \text{ even}\}$, $S^- = \{A \in S \mid w(A) \text{ odd}\}$. Find an alternating involution to show that
$$|S^+| - |S^-| = \begin{cases} 0, & k \text{ odd;} \\ (-1)^{k/2}\binom{n}{k/2}, & k \text{ even.} \end{cases}$$

9. (10%) Show that any permutation of $\{1, 2, \dots, mn + 1\}$ contains an increasing subword of length $m + 1$ or a decreasing subword of length $n + 1$.