

You only need to choose 8 problems to answer.

1. Let  $G$  be a group and  $N, H$  be subgroups of  $G$ . Suppose that  $N \triangleleft G$ ,  $|H|$  is finite and  $[G : N]$  is finite. If  $[G : N]$  and  $|H|$  are relatively prime, show that  $H$  is contained in  $N$ .
2. Let  $G$  be a group of order 2012.
  - (a) Find the number of subgroups of order 503.
  - (b) Find the number of elements of order 503.
3. Let  $R$  be a principal ideal domain and let  $J$  be a nonzero ideal of  $R$ . Show that  $J$  is a maximal ideal of  $R$  if and only if  $J$  is a prime ideal of  $R$ .
4. Let  $\mathbb{Z}$  be the ring of integers and  $\mathbb{Q}$  the additive group of rational numbers. Show that  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$  as groups.
5. Let  $f(x) = x^4 - 2 \in \mathbb{Q}[x]$  and let  $u = \sqrt[4]{2}$  be the positive real fourth root of 2. Suppose that  $F \subseteq \mathbb{C}$  is a splitting field of  $f(x)$  over  $\mathbb{Q}$ .
  - (a) Show that  $f(x)$  is irreducible over  $\mathbb{Q}$ .
  - (b) Is  $\mathbb{Q}(u)$  normal over  $\mathbb{Q}$ ?
  - (c) Find the order of the Galois group  $\text{Aut}_{\mathbb{Q}}F$ .
6. Let  $G$  be a group and let  $n \in \mathbb{N}$ . Suppose  $H$  is the only subgroup of  $G$  of order  $n$ . Prove that  $H$  is a normal subgroup of  $G$ .
7. Let  $G$  be a finitely generated abelian group in which no element, except 0, has finite order. Prove that  $G$  is a free abelian group.
8. Let  $I$  be an ideal in a commutative ring  $R$ . Let  $\text{Rad}I = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}$ . Prove that  $\text{Rad}I$  is an ideal of  $R$ .
9. Let  $R$  be a ring with identity and suppose  $f : A \rightarrow B$  and  $g : B \rightarrow A$  are  $R$ -module homomorphisms such that  $gf = 1_A$ . Prove that  $B = \text{Im}f \oplus \text{Ker}g$ , i.e.,  $B = \text{Im}f + \text{Ker}g$  and  $\text{Im}f \cap \text{Ker}g = 0$ .
10. Let  $F$  be an extension field of a field  $K$  and let  $u, v \in F$  be algebraic over  $K$ . Suppose the irreducible polynomials of  $u$  and  $v$  over  $K$  have degree  $m$  and  $n$  respectively.
  - (a) Prove that  $[K(u, v) : K] \leq mn$ .
  - (b) If we further assume that  $m$  and  $n$  are relatively prime, prove that  $[K(u, v) : K] = mn$ .

# 國立台灣師範大學數學系博士班資格考試

科目：代數

2013年4月30日

1. (16 pts) Let  $f : G \rightarrow H$  be a group homomorphism.
  - (a) Suppose  $a \in G$  has finite order  $n$ . Prove that  $f(a) \in H$  has finite order  $m$  with  $m \mid n$ .
  - (b) Suppose  $G$  is cyclic and  $f$  is onto. Prove that  $H$  is also cyclic.
2. (18 pts) Let  $G$  be a finite group with  $|G| = p^n q$  ( $n \geq 1$ ) where  $p, q$  are primes such that  $p > q$ .
  - (a) Show that  $G$  is not a simple group.
  - (b) Assume that  $G$  acts on a set  $X$  with  $|X| = q$ . Show that this action must be either trivial or transitive. (Recall that  $G$  acts on  $X$  trivially if  $g \cdot x = x$  for all  $x \in X$  and all  $g \in G$  and the action is transitive if for any  $x_1, x_2 \in X$  there exists a  $g \in G$  such that  $x_2 = g \cdot x_1$ .)
3. Let  $R$  be a commutative ring with identity 1.
  - (a) (10 pts) Let  $M$  be an ideal of  $R$ . Prove that  $M$  is a maximal ideal if and only if for every  $r \in R \setminus M$ , there exists  $x \in R$  such that  $1 - rx \in M$ .
  - (b) (12 pts) Let  $J$  be the intersection of all maximal ideals of  $R$  and let  $U(R)$  be the group of units of  $R$ . Prove that  $1 + J = \{1 + x \mid x \in J\}$  is a subgroup of  $U(R)$ .
4. (12 pts) Let  $R$  be a principal ideal domain and let  $B$  be a submodule of a unitary  $R$ -module  $A$ . Suppose  $A$  can be generated by  $n$  elements with  $n < \infty$ . Prove that  $B$  can be generated by  $m$  elements with  $m \leq n$ .
5. (14 pts)
  - (a) Construct a finite field of 125 elements. Does there also exist a finite field of 120 elements? (You need to explain your answer.)
  - (b) Let  $\mathbb{E}$  be a finite extension of a finite field  $\mathbb{F}$ . Show that  $\mathbb{E}$  must be a Galois extension of  $\mathbb{F}$  such that the Galois group  $\text{Aut}_{\mathbb{F}}(\mathbb{E})$  of  $\mathbb{E}$  over  $\mathbb{F}$  is a cyclic group.
6. (18 pts) Let  $K = \mathbb{C}(t)$  be the rational function field in the variable  $t$  over the complex numbers  $\mathbb{C}$ . Let  $n$  be a positive integer and let  $f(x) = x^n + t \in K[x]$ .
  - (a) Prove or disprove that  $f(x)$  is irreducible over  $K$ .
  - (b) Let  $\bar{K}$  be an algebraic closure of  $K$  and let  $u \in \bar{K}$  be a zero of  $f(x)$ . Let  $L = K(u)$ . Show that  $L$  is Galois over  $K$  and that for every divisor  $d$  of  $n$  there exists a unique intermediate subfield  $M$  of  $L$  (i.e.  $K \subseteq M \subseteq L$ ) such that  $[M : K] = d$ .

# Algebra Qualifying Exam

Fall 2013

- Please choose **Five** of the following six questions to answer.

- (a) Let  $G$  be a group and suppose that  $H$  is a normal subgroup of  $G$ . If  $H$  is cyclic, prove that every subgroup of  $H$  is normal in  $G$ .
  - (b) Find a finite group  $G$  which has subgroups  $H$  and  $K$  satisfying the following conditions:
    - i.  $H$  is a normal subgroup of  $K$ .
    - ii.  $K$  is a normal subgroup of  $G$ .
    - iii.  $H$  is not a normal subgroup of  $G$ .
- Let  $G$  be a group of order 2013.
  - (a) Show that  $G$  has a normal subgroup of order 11.
  - (b) Show that  $G$  has a subgroup of order 33 and such a subgroup is abelian.
- Let  $R$  be a ring with identity 1. Recall that an ideal  $P$  in  $R$  is said to be prime if  $P \neq R$  and for any ideals  $A, B$  in  $R$ , we have  $AB \subseteq P$  implies  $A \subseteq P$  or  $B \subseteq P$ . Now suppose that  $I$  is an ideal of  $R$  and  $I \neq R$ . Show that the following are equivalent.
  - (a)  $I$  is a prime ideal of  $R$ .
  - (b) If  $r, s \in R$  such that  $rRs \subseteq I$ , then  $r \in I$  or  $s \in I$ .
- Consider  $\mathbb{Q}$  as a  $\mathbb{Z}$ -module.
  - (a) Prove that any two distinct elements  $\alpha, \beta \in \mathbb{Q} \setminus \{0\}$  are linearly dependent over  $\mathbb{Z}$ .
  - (b) Prove that no element in  $\mathbb{Q}$  can generate  $\mathbb{Q}$  over  $\mathbb{Z}$ , i.e., for any  $q \in \mathbb{Q}$ ,  $\langle q \rangle \subsetneq \mathbb{Q}$  where  $\langle q \rangle = \{nq \mid n \in \mathbb{Z}\}$ .
  - (c) Prove that  $\mathbb{Q}$  is not a free  $\mathbb{Z}$ -module.
- (a) Prove that  $x^3 + 2x + 1 \in \mathbb{Z}_7[x]$  is an irreducible polynomial in  $\mathbb{Z}_7[x]$ .
  - (b) Construct a field of 27 elements.
  - (c) Is there a field of 2013 elements? Explain your answer.
- (a) Let  $u \in \mathbb{C}$  be a zero of the polynomial  $x^4 + 2x + 2$ . Please write  $\frac{1}{u}$  as a polynomial of  $u$ , i.e., find a polynomial  $f(x) \in \mathbb{Q}[x]$  such that  $\frac{1}{u} = f(u)$ .
  - (b) Let  $K$  be an algebraic extension field of a field  $F$  and let  $D$  be an integral domain such that  $F \subseteq D \subseteq K$ . Prove that  $D$  is indeed a field.

# Algebra Qualifying Exam

Spring 2014

1. Show that every finitely generated subgroup of the additive group  $\mathbb{Q}$  is cyclic. (8 pts)
2. (a) Define the characteristic of a ring. (4 pts)  
(b) Let  $F$  be a field. Show that the characteristic of  $F$  is either 0 or a prime  $p$ . (8 pts)  
(c) Let  $F$  be a finite field of prime characteristic  $p$ . Show that  $F$  has  $p^n$  elements for some positive integer  $n$ . (8 pts)
3. Show that a group of order  $p^2q$ , where  $p$  and  $q$  are distinct primes, contains a normal Sylow subgroup. (12 pts)
4. Let  $R = \mathbb{Z}/7\mathbb{Z}$ .  
(a) Show that the polynomial ring  $R[x]$  is a principal ideal domain. (8 pts)  
(b) Find all prime ideals of the ring  $R[x]/\langle x^2 - 2 \rangle$ . (8 pts)
5. Let  $F = \mathbb{Q}(\sqrt{3}, \sqrt{11})$ .  
(a) Find the Galois group  $\text{Aut}_{\mathbb{Q}}F$ . (6 pts)  
(b) Find the corresponding intermediate fields of  $F$ . (6 pts)  
(c) Find all normal extensions of  $\mathbb{Q}$  in  $F$ . (6 pts)
6. Show that any ring with identity is isomorphic to a ring of endomorphisms of an abelian group. (8 pts)
7. Let  $R$  be a commutative ring with identity and let  $M$  be a finitely generated  $R$ -module. Let  $f : M \rightarrow R^n$  be a surjective  $R$ -module homomorphism. Show that  $\text{Ker } f$  is finitely generated. (8 pts)
8. Let  $G$  be a group and let  $C(G) = \{a \in G \mid ga = ag, \forall g \in G\}$ .  
(a) Prove that  $C(G)$  is a normal subgroup of  $G$ . (8 pts)  
(b) Prove that if  $G/C(G)$  is cyclic then  $G$  is abelian. (8 pts)
9. Let  $R$  be a commutative ring with identity and let  $I = \{r \in R \mid r^n = 0 \text{ for some } n \in \mathbb{N}\}$ .  
(a) Prove that  $I$  is an ideal of  $R$ . (6 pts)  
(b) Prove that every prime ideal of  $R$  contains  $I$ . (6 pts)
10. Let  $R$  be a ring with identity and let  $M$  be an  $R$ -module. Suppose  $f : M \rightarrow M$  is an  $R$ -module homomorphism such that  $ff = f$ . Prove that  $M = \text{Ker } f \oplus \text{Im } f$ . (8 pts)
11. Suppose  $R$  is a commutative ring with identity such that every submodule of every free  $R$ -module is free. Prove that  $R$  is a principal ideal domain. (12 pts)
12. Prove that no finite field is algebraically closed. (8 pts)

# Algebra Qualifying Exam

Fall 2015

1. Let  $G$  be an abelian group and let  $a, b \in G$ . Suppose the order of  $a$  is  $m$  and the order of  $b$  is  $n$  and  $\gcd(m, n) = 1$ . Prove that the order of  $ab$  is  $mn$ . (10 pts)
2. Let  $G$  be a group of order 63.
  - (a) Prove that  $G$  is not simple. (6 pts)
  - (b) Prove that  $G$  contains a subgroup of order 21. (8 pts)
3. Prove that  $\mathbb{Q}$  is not a free abelian group, i.e., not a free  $\mathbb{Z}$ -module. (12 pts)
4. Let  $R$  and  $S$  be commutative rings with  $1 \neq 0$  and let  $f : R \rightarrow S$  be a homomorphism of commutative rings. Suppose  $J$  is an ideal of  $S$ .
  - (a) Prove that  $f^{-1}(J) = \{a \in R \mid f(a) \in J\}$  is an ideal of  $R$ . (5 pts)
  - (b) Assume  $f(1_R) = 1_S$  and  $J$  is a prime ideal of  $S$ . Prove that  $f^{-1}(J)$  is a prime ideal of  $R$ . (7 pts)
5. Let  $R$  be a commutative ring with identity  $1 \neq 0$  and let  $J$  be the intersection of all maximal ideals of  $R$ . Consider an element  $x \in R$ .
  - (a) Suppose  $x \in J$ . Prove that  $1 + x$  is a unit in  $R$ . (6 pts)
  - (b) Suppose  $x \notin J$ . Prove that there exists an element  $r \in R$  such that  $1 - rx$  is not a unit in  $R$ . (8 pts)
6. Let  $R$  be an integral domain and let  $A$  be a unitary  $R$ -module. Prove that
$$T(A) = \{a \in A \mid ra = 0 \text{ for some } \textit{nonzero } r \in R\}$$
is a submodule of  $A$ . (8 pts)
7. Let  $R$  be a principal ideal domain and let  $A$  be a finitely generated unitary  $R$ -module. Suppose  $A$  can be generated by  $n$  elements and let  $B$  be a submodule of  $A$ . Prove that  $B$  can be generated by  $m$  elements with  $m \leq n$ . (10 pts)
8. Prove that  $\mathbb{Q}[i]$  and  $\mathbb{Q}[\sqrt{2}]$  are isomorphic as  $\mathbb{Q}$ -vector spaces but they are not isomorphic as fields. (8 pts)
9. Let  $K \leq E \leq F$  be fields and suppose  $F$  is a cyclic extension of  $K$ , i.e.,  $F$  is *algebraic and Galois over  $K$  and the Galois group  $\text{Aut}_K F$  is cyclic*. Prove that  $F$  is a cyclic extension of  $E$  and  $E$  is a cyclic extension of  $K$ . (12 pts)

# Algebra Qualifying Exam

Spring 2016

- (a) Please state Lagrange's Theorem. (4 pts)  
(b) Please state Cauchy's Theorem. (4 pts)  
(c) Please state the First Isomorphism Theorem. (4 pts)
- Let  $G_1$  and  $G_2$  be groups. Suppose  $N_1$  is a normal subgroup of  $G_1$  and  $N_2$  is a normal subgroup of  $G_2$ . Prove that  $N_1 \times N_2$  is a normal subgroup of  $G_1 \times G_2$  and

$$(G_1 \times G_2)/(N_1 \times N_2) \simeq G_1/N_1 \times G_2/N_2. \quad (12 \text{ pts})$$

- Let  $p > q$  be two prime numbers and suppose  $G$  is a group of order  $p^2q$ .  
(a) Prove that  $G$  is not simple. (7 pts)  
(b) Prove that  $G$  contains at least three cyclic subgroups. (7 pts)
- Suppose  $R$  is a ring such that  $r^2 = r$  for all  $r \in R$ . Prove that  $R$  is commutative. (8 pts)
- Let  $R$  be a commutative rings with  $1 \neq 0$  and let  $J = \{r \in R \mid r^n = 0 \text{ for some } n \in \mathbb{N}\}$ .  
(a) Prove that  $J$  is an ideal of  $R$ . (7 pts)  
(b) Suppose  $P$  is a prime ideal of  $R$ . Prove that  $J \subseteq P$ . (7 pts)
- (a) Let  $R$  be a commutative ring and let  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be a sequence of  $R$ -modules and  $R$ -module homomorphisms. What does it mean that this sequence is exact? (7 pts)  
(b) Suppose  $0 \rightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} V_4 \xrightarrow{f_4} V_5 \rightarrow 0$  is an exact sequence of  $\mathbb{Q}$ -vector spaces and linear transformations. Prove that

$$\dim V_1 + \dim V_3 + \dim V_5 = \dim V_2 + \dim V_4. \quad (7 \text{ pts})$$

- Let  $F$  be an algebraically closed field. Prove that  $|F| = \infty$ . (8 pts)
- Let  $F$  be an extension field of a field  $K$ .  
(a) What does it mean that  $F$  is normal over  $K$ ? (6 pts)  
(b) Is  $\mathbb{Q}(\sqrt[4]{2})$  normal over  $\mathbb{Q}$ ? Please explain your answer. (6 pts)  
(c) Is  $\mathbb{Q}(\sqrt[4]{2}, i)$  normal over  $\mathbb{Q}$ ? Please explain your answer. (6 pts)

**Algebra Qualifying Exam**  
**Fall 2018**

1. Let  $G$  be a simple group of order 168.
  - (a) (8 pts) Find the number of elements of order 7 in  $G$ .
  - (b) (6 pts) Suppose that  $P$  is a Sylow 7-subgroup of  $G$  and  $\mathbf{N}_G(P)$  is the normalizer of  $P$  in  $G$ . Find the order of  $\mathbf{N}_G(P)$ .
  - (c) (8 pts) Prove that  $G$  has no element of order 14.
2. Let  $G$  be a group. Recall that for each  $g \in G$ , a map  $\theta_g : G \rightarrow G$  given by  $\theta_g(x) = gxg^{-1}$  is called an inner automorphism of  $G$ . Let  $\text{Inn}(G)$  be the group of all inner automorphisms of  $G$ .
  - (a) (8 pts) Let  $\mathbf{Z}(G)$  be the center of  $G$ . Prove that  $G/\mathbf{Z}(G) \cong \text{Inn}(G)$ .
  - (b) (8 pts) Let  $S_4$  be the symmetric group of degree 4. Show that  $S_4 \cong \text{Inn}(S_4)$ .
3. (10 pts) If  $R$  is a principal ideal domain, is it always true that the polynomial ring  $R[x]$  is a principal ideal domain? Justify your answer.
4. (10 pts) Let  $R$  be a ring with 1 and let  $M$  be a unitary  $R$ -module. Suppose  $f : M \rightarrow M$  is an  $R$ -module homomorphism such that  $ff = f$ . Prove that  $M = \text{Ker } f \oplus \text{Im } f$ .
5. (10 pts) Let  $R$  be a principal ideal domain and let  $A$  be a finitely generated unitary  $R$ -module. Suppose  $A$  can be generated by  $n$  elements and let  $B$  be a submodule of  $A$ . Prove that  $B$  can be generated by  $m$  elements with  $m \leq n$ .
6. Let  $\mathbb{Q}$  be the field of rational numbers and let  $\mathbb{C}$  be the field of complex numbers. Let  $f(x) = x^4 - 5$  in  $\mathbb{Q}[x]$ . Suppose that  $E \subseteq \mathbb{C}$  is the splitting field of  $f(x)$  over  $\mathbb{Q}$ .
  - (a) (8 pts) Show that  $f(x)$  is irreducible over  $\mathbb{Q}$ .
  - (b) (8 pts) Let  $\alpha = \sqrt[4]{5}$  be the unique positive real root of  $x^4 - 5$ . Let  $i = \sqrt{-1}$  in  $\mathbb{C}$ . Show that  $E = \mathbb{Q}(\alpha, i)$ .
  - (c) (8 pts) Determine  $[E : \mathbb{Q}]$ .
  - (d) (8 pts) Let  $K = \mathbb{Q}(\sqrt{5})$ . Determine the Galois group  $\text{Aut}_K E$ .

**Algebra Qualifying Exam.**  
**Spring 2019**

1. (a) Let  $F$  be a field and let  $F^* = F - \{0\}$  be the multiplicative group of  $F$ . Show that every finite subgroup of  $F^*$  is cyclic. (10 pts)  
(b) Describe all finite subgroups of  $C^* = C - \{0\}$ , where  $C$  is the field of complex numbers. (5 pts)
2. (a) Let  $f(x) \in Q[x]$  with degree  $n$ . Show that if  $f(x)$  is irreducible over  $Q$ , then the Galois group  $G_f$  of  $f(x)$  over  $Q$  is a transitive subgroup of  $S_n$ . (8 pts)  
(b) For  $f(x) = x^5 - 6x + 3$ , show that the Galois group  $G_f$  of  $f(x)$  over  $Q$  is  $S_5$ . (7 pts)
3. Let  $R$  be a ring and  $J$  be an ideal of  $R$ .  
(a) Show that  $M_{2 \times 2}(J)$  is an ideal of  $M_{2 \times 2}(R)$ . (5 pts)  
(b) Show that every ideal of  $M_{2 \times 2}(R)$  is of the form  $M_{2 \times 2}(J)$ , where  $J$  is an ideal of  $R$ . (12 pts)
4. (a) Find the ideal consists of all the nilpotent elements of the ring  $Z_{12}$ . (5 pts)  
(b) For a commutative ring  $R$  with  $1$ , show that the ideal of  $R$  which consists of all the nilpotent elements of  $R$  is equal to the intersection of all prime ideals of  $R$ . (12 pts)
5. Let  $G$  be a nonabelian group of order 12 which has a normal subgroup of order 4.  
(a) Show that  $G$  has 4 Sylow 3-subgroups. (6 pts)  
(b) Show that  $G$  is isomorphic to the Alternating group  $A_4$ . (10pts)
6. Let  $G$  be a finite group and let  $p$  be the smallest prime divisor of the order of  $G$ . Show that every subgroup  $H$  of  $G$  of index  $p$  is a normal subgroup of  $G$ . (10 pts)
7. Let  $R$  be a ring with identity. Suppose  $M_1, M_2$  and  $N$  are submodules of an  $R$ -module  $M$  such that  $M_1$  is a submodule of  $M_2$ . Show that there is an exact sequence of  $R$ -modules  
$$0 \rightarrow (M_2 \cap N)/(M_1 \cap N) \rightarrow M_2/M_1 \rightarrow (M_2 + N)/(M_1 + N) \rightarrow 0. \quad (10 \text{ pts})$$

**Notations :**  $Q$  : The field of rational numbers.  $M_{2 \times 2}(R)$  :  $2 \times 2$  matrix ring over  $R$ .



## Algebra Qualifying Exam

Fall 2019

- Let  $S_n$  be the symmetric group of degree  $n$ . Let  $A_n$  be the alternating group of degree  $n$ .
  - Find the number of elements of order 3 in  $S_5$ . (6 pts)
  - Let  $P$  be a Sylow 3-subgroup of  $S_5$ . Let  $N$  be the normalizer of  $P$  in  $S_5$ . Find the order of  $N$ . (6 pts)
  - Find the number of Sylow 2-subgroups of  $S_4$ . (6 pts)
  - Show that  $S_4$  is solvable. (8 pts)
  - Is it true that every finite group is isomorphic to a subgroup of  $A_n$  for some positive integer  $n$ ? (8 pts)
- Let  $R$  be a ring with identity. An element  $r$  in  $R$  is called nilpotent if  $r^n = 0$  for some positive integer  $n$ . An ideal  $I$  of  $R$  is called nil if every element of  $I$  is nilpotent. Suppose that  $R$  is not commutative. For any two nil ideals  $J, K$  of  $R$ , is it always true that  $J + K$  is nil? (10 pts)
- If  $R$  is a unique factorization domain, show that every nonzero prime ideal in  $R$  contains a nonzero principal ideal that is prime. (10 pts)
- Let  $R$  be a ring with identity. Let  $A, B$  be  $R$ -modules. Let  $1_A$  be the identity function on  $A$ . Suppose  $f : A \rightarrow B$  and  $g : B \rightarrow A$  are  $R$ -module homomorphisms such that  $gf = 1_A$ . Prove that  $B = \text{Im} f \oplus \text{Ker} g$ . (10 pts)
- Let  $K$  be a field and let  $f(x)$  be a polynomial in  $K[x]$  of positive degree. Let  $E$  be a splitting field of  $f(x)$  over  $K$  and let  $G = \text{Aut}_K(E)$  be the Galois group of  $f$ . Let  $\Lambda = \{ \alpha \in E \mid f(\alpha) = 0 \}$ . We use  $\text{Sym}(\Lambda)$  to denote the group of all bijections on  $\Lambda$  under the operation of function composition.
  - Show that  $G$  is isomorphic to a subgroup of  $\text{Sym}(\Lambda)$ . (8 pts)
  - If  $f(x)$  is irreducible separable over  $K$  and  $f(x)$  has degree  $n$ , show that  $n$  divides  $|G|$  and  $G$  is isomorphic to a transitive subgroup of  $S_n$ . (10 pts)
- Let  $f(x) = x^4 + 2x^2 + 4 \in \mathbb{Q}[x]$ .
  - Show that  $f(x)$  is irreducible over  $\mathbb{Q}$ . (8 pts)
  - Let  $E$  be the splitting field of  $f(x)$  over  $\mathbb{Q}$  in  $\mathbb{C}$ . Determine the Galois group  $\text{Aut}_{\mathbb{Q}} E$  of  $f(x)$  over  $\mathbb{Q}$ . (10 pts)

## Algebra Qualifying Exam

### Spring 2020

- Let  $\mathbb{Q}$  be the field of rational numbers.
  - Let  $\mathbb{R}$  be the field of real numbers.
1. (20 pts) Let  $G$  be a group. Let  $G'$  be the commutator subgroup of  $G$ . Let  $G''$  be the commutator subgroup of  $G'$ .
    - (a) If  $H$  is a subgroup of  $G$ , show that  $N_G(H)/C_G(H)$  is isomorphic to a subgroup of  $\text{Aut}(H)$ .
    - (b) If  $G''$  is cyclic, show that  $G''$  is contained in the center of  $G'$ .
  2. (10 pts) Let  $D$  be the dihedral group of order  $2n$ . Write  $2n = 2^k m$  where  $k, m$  are positive integers and  $m$  is odd. Show that  $D$  has exactly  $m$  Sylow 2-subgroups.
  3. (20 pts) Let  $R$  be a ring with 1 and let  $M_2(R)$  be the ring of  $2 \times 2$  matrices over  $R$ . If  $I$  is an ideal of  $R$ , we know that  $M_2(I)$  is an ideal of  $M_2(R)$ .
    - (a) Show that if  $J$  is an ideal of  $M_2(R)$ , then  $J = M_2(I)$  for some ideal  $I$  of  $R$ .
    - (b) If  $L$  is a maximal ideal of  $R$ , is it always true that  $M_2(L)$  is a maximal ideal of  $M_2(R)$ ?
  4. (20 pts) Let  $R$  be a ring and let

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \xrightarrow{h} & A_4 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\ B_1 & \xrightarrow{f'} & B_2 & \xrightarrow{g'} & B_3 & \xrightarrow{h'} & B_4 \end{array}$$

be a commutative diagram of  $R$ -modules and  $R$ -module homomorphisms with exact rows. Suppose that  $\alpha$  is an epimorphism and  $\delta$  is a monomorphism.

- (a) If  $\beta$  is a monomorphism, show that  $\gamma$  is a monomorphism.
  - (b) If  $\gamma$  is an epimorphism, show that  $\beta$  is an epimorphism.
5. (15 pts)
  - (a) Suppose  $\sigma \in \text{Aut}_{\mathbb{Q}}(\mathbb{R})$  and  $r \in \mathbb{R}$ . If  $r > 0$ , show that  $\sigma(r) > 0$ .
  - (b) Prove that  $\text{Aut}_{\mathbb{Q}}(\mathbb{R})$  is the trivial group.
6. (15 pts) Let  $f(x) = (x^3 - 2)(x^2 + 3) \in \mathbb{Q}[x]$ . Let  $K$  be a splitting field of  $f(x)$  over  $\mathbb{Q}$ . Determine the Galois group  $\text{Aut}_{\mathbb{Q}}(K)$  of  $f(x)$ .