國立台灣師範大學數學系
103 學年度下學期博士班資格考試題
科目：微分方程/PDEs

Time and Date: 9-12, April 22, 2015

1. (10 pt.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u = e^{x+2t}$$

with $u(x,0) = 0$. Find the explicit solution of this equation.

2. (10 pt.) Solve the wave equation of $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^{2x},\ \forall\ (x,t) \in \mathbb{R}^2,$$

with the conditions, $u(x,0) = 0 = u(0,t)$, $\forall\, t, x \in \mathbb{R}$.

3. (20 pt.) Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a real-valued function, and

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy.$$

Show with details that

(a)

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1,\ \forall\, t > 0.$$

(b) For any fixed $x \in \mathbb{R}^n$,

$$\lim_{t \to +0} u(x,t) = g(x).$$

4. (10 pt.) Let $\mathbb{R}^2_+$ be the set of upper half-plane. Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be a harmonic function satisfying $u(x,y) = f(x/y)$, and the boundary conditions $u(x,0) = 1$ for $x > 0$ and $u(x,0) = 0$ for $x < 0$. Find the explicit formula of the solution $u(x,y)$.

5. (30 pt.) Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth harmonic function, i.e. $\Delta u = 0$. Denote the gradient of u by ∇u, and the Hessian (matrix) of u by $\nabla^2 u$.

   (a) Show that $|\nabla u|^2$ is subharmonic, i.e. $\Delta |\nabla u|^2 \geq 0$.

   (b) Show that for any $k \in [0, n]$,

$$\frac{d}{dr}\left(\frac{1}{r^k}\int_{\mathbb{B}_r(0)} |\nabla u|^2 \, dx\right) \geq 0$$

   where $\mathbb{B}_r(0) := \{x \in \mathbb{R}^n : |x| < r\}$.

   (c) Let $n = 2$, and assume that $\det \nabla^2 u \neq 0$ at a point $p \in \mathbb{R}^2$. Show that $\det \nabla^2 u$ is superharmonic (i.e. $\Delta \det \nabla^2 u \leq 0$) in a neighborhood of p.

6. (20 pt.) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded simply-connected subset.

   (a) Give the definition of Sobolev spaces $W^{1,2}(\Omega)$ is in terms of the notion of weak derivatives. Is $W^{1,2}(\Omega)$ a Hilbert space (explain your answer)?

   (b) Let $p \in [1, n)$. If we want to establish an estimate of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

   for any function $u \in C_c^\infty(\mathbb{R}^n)$ and certain constants $C > 0$, $q \in [1, \infty)$, what should the algebraic relation of p, q, and n be?

   (Hint: scaling of u in either the domaun or the range would provide the information)

國立台灣師範大學數學系
104 學年度上學期博士班資格考試題
科目：偏微分方程
Math/NTNU Qualifying Exam of PDEs in Oct. 2015

Time and Date: 2-5 PM, October 31, 2015

1. (15 pt.)

   Let $u : [0,\pi] \times (\mathbb{R}_+ \cup \{0\}) \to \mathbb{R}$ fulfill

   $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \ \forall\ (x,t) \in (0,\pi) \times \mathbb{R}_+$$

   with the initial data

   $$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin nx, \quad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \beta_n \sin nx,$$

   and boundary conditions

   $$u(0,t) = 0 = u(\pi,t), \quad \forall\ t > 0.$$

   Represent the solution u as a Fourier series

   $$u(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) \sin nx,$$

   and compute the coefficients $\gamma_n(t)$.

2. (25 pt.)

   Let

   $$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4\ t}}, \forall\ (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

Assume that there exist constant numbers, $M > 0$ and $\alpha \in (0,1)$, such that the real-valued function $f \in C(\mathbb{R}\times\mathbb{R}) \cap L^\infty(\mathbb{R}\times\mathbb{R})$ fulfills

$$|f(x_2,t_2) - f(x_1,t_1)| \le M \cdot (|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2})$$

for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R}^2$. Let

$$z(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-y, t-\tau) \cdot f(y,\tau)\, dy\, d\tau.$$

Show with details that

(a) (20 pt.) $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x^2}$ are continuous in $\mathbb{R}\times(\mathbb{R}_+ \cup \{0\})$.

(b) (5 pt.) The equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f$$

holds in the domain $\mathbb{R}\times\mathbb{R}_+$.

Hint: Consider the family of functions,

$$z_h(x,t) = \int_0^{t-h} \int_{-\infty}^{\infty} K(x-y, t-\tau) \cdot f(y,\tau)\, dy\, d\tau,$$

where $h \in (0, t/2)$.

3. (30 pt.) The following is a standard procedure to establish regularity of weak solutions of elliptic PDEs.

   Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, and simply-connected subset. Assume that $\Gamma : \mathbb{R} \to \mathbb{R}$ is smooth and bounded (i.e., $|\Gamma| \le M$ for some constant $M > 0$). Suppose that $v : \Omega \to \mathbb{R}$ is a weak solution of

$$\Delta u + \Gamma(u)|\nabla u|^2 = 0 \tag{1}$$

   in the Sobolev space $W^{1,2}(\Omega)$. If v is a weak solution of Eq.(1) in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$, where $p > d$, then the L^p-theory of elliptic PDEs implies

$$v \in W^{2,q}(\Omega')$$

   for some $q \in (1,\infty)$ and any proper open set $\Omega' \subset \Omega$. The so-called boot-strapping argument is to proceed this procedure until one derives the interior smoothness of v, i.e. $v \in C^\infty(\Omega)$.

(a) (10 pt.) Give the definitions of weak derivatives and weak solutions of Eq.(1) in $W^{1,2}(\Omega)$.

(b) (20 pt.) Explain how to apply the boot-strapping argument to derive interior smoothness of v, i.e. prove that $v \in C^\infty(\Omega)$.

Hints: You should first figure out `$q = ?$' in each step stated above. In other words, in the L^p-theory, $\Delta v = f \in L^r$ for some $r > 1$ implies that $v \in W^{2,s}(\Omega')$, where $s = ?$

4. (30 pt.) Denote by $\mathbb{B}_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^d$. Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$u(x) = \frac{x}{|x|}.$$

(a) As $d = 1$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1, \infty)$? If it is yes, what is the range of p? If it is not, explain why.

(b) As $d = 2$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1, \infty)$? If it is yes, what is the range of p? If it is not, explain why.

(c) As $d = 3$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1, \infty)$? If it is yes, what is the range of p? If it is not, explain why.

國立台灣師範大學數學系
104 學年度下學期博士班資格考試題
科目：偏微分方程
Math/NTNU Qualify Exam of PDEs on April 30, 2016

Time and Date: 3 hours, April 30, 2016

總共 5 大題，滿分 110 分。

1. (20 pt.) Denote by $\mathbb{U}_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^d$. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$u(x) = \frac{x_1}{\sqrt{\sum_{j=1}^{d} x_j^2}}.$$

   (a) As $d = 1$, is it true that $u \in W^{1,p}(\mathbb{U}_1)$ for some $p \in [1,\infty)$? If the answer is positive, what is the range of p? If it is negative, explain why.

   (b) As $d = 2$, is it true that $u \in W^{1,p}(\mathbb{U}_1)$ for some $p \in [1,\infty)$? If the answer is positive, what is the range of p? If it is negative, explain why.

2. (20 pt.) Denote by $C_c^\infty(\mathbb{R}^d)$ the class of smooth real-valued functions with compact support in $\mathbb{R}^d$.

   (a) Show that any function $u \in C_c^\infty(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} (\Delta u)^2\, dx = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 dx, \tag{1}$$

   where Δ denotes the Laplace operator in $\mathbb{R}^d$.

   (b) Explain why Eq.(1) also holds for any function $u \in C_c^2(\mathbb{R}^d)$.

3. (10 pt.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u = e^{2x-t}$$

with $u(x,0) = 0$. Find the explicit solution of this equation.

4. (30 pt.) Denote by $\mathbb{U}_R := \{x \in \mathbb{R}^2 : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^2$, and by $\mathbb{B}_R := \{x \in \mathbb{R}^2 : |x| \leq R\}$ the closed ball of radius $R > 0$ with center at the origin of $\mathbb{R}^2$. Suppose the functions g and $u : \mathbb{B}_R \to \mathbb{R}$ are continuous and u satisfies the Poisson equation,

$$\Delta u = 0, \text{ in } \mathbb{U}_R,$$

with boundary value g, i.e.

$$\lim_{x \to x_0} u(x) = g(x_0), \; \forall \, x_0 \in \partial \mathbb{B}_R.$$

Answer the following questions with sufficient details.

(a) The fundamental solution of Laplace equation is given by

$$\Gamma(x,y) = \frac{1}{2\pi} \log |x - y|,$$

where $x, y \in \mathbb{R}^2$. Show that the Poisson representation formula is given by

$$u(x) = \frac{R^2 - |x|^2}{2\pi R} \int_{y \in \partial \mathbb{B}_R} \frac{g(y)}{|x-y|^2} \, do(y), \; \forall \, x \in \mathbb{U}_R,$$

where $do(y)$ represents the arclength element of $\partial \mathbb{B}_R$ at y. (Hint: you might need Schwartz reflection principle to construct the so-called Green's functions and apply Green's identity.)

(b) Show that

$$\lim_{x \to x_0} u(x) = g(x_0),$$

for any $x_0 \in \partial \mathbb{B}_R$.

(c) There are several methods to prove Maximum Principle for harmonic functions. Could you just use the Poisson representation formula to prove the strong Maximum Principle of the harmonic function u? Namely, if

$$\sup_{\mathbb{B}_R} u = u(p), \text{ for some } p \in \mathbb{U}_R,$$

then u is a constant function.

5. (30 pt.) Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbb{R}$ and $t > 0$. Let $g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a real-valued function, and

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) g(y)\, dy.$$

Show with details that

(a) the function u satisfies the heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

in $\mathbb{R} \times \mathbb{R}_+$.

(b) For any fixed $x \in \mathbb{R}$,

$$\lim_{t \to +0} u(x,t) = g(x).$$

(c) if

$$\int_{\mathbb{R}} |g(x)|^2\, dx \leq M,$$

for some constant $M > 0$, then there exists a constant C such that

$$|u(x,t)| \leq \frac{C}{t^{1/4}},$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$,

PDE Qualify Exam

2016/10/31

1. Solve following problems. (10 points for each problem)

(1). $$\begin{cases} \dfrac{1}{(1+x)^2}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 \text{ for } 0 < x < 1,\ t > 0; \\ u(0,t) = 0; \\ u(1,t) = 0; \\ u(x,0) = 0; \\ \dfrac{\partial u}{\partial t}(x,0) = g(x). \end{cases}$$

(2). $$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 \text{ for } r < 1; \\ u(1,\theta) = \sin^2\theta. \end{cases}$$

(3). $$\begin{cases} \Delta u - u = 0 \text{ for } 0 < x < \pi,\ 0 < y < \pi/2,\ 0 < z < 1; \\ u = 0 \quad \text{for } x = 0,\ y = 0,\ z = 1; \\ \dfrac{\partial u}{\partial x} = 0 \quad \text{for } x = \pi; \\ \dfrac{\partial u}{\partial x} = 0 \quad \text{for } y = \dfrac{\pi}{2}; \\ \dfrac{\partial u}{\partial z}(x,y,0) = 2x - \pi. \end{cases}$$

2. (a). Show that if $\begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = 0 \text{ for } 0 < x < l; \\ \dfrac{\partial u}{\partial x}(0,t) = 0. \end{cases}$ the maximum of u for $0 \le x \le l$ and $0 \le t \le t_1$ must occur at $t = 0$ or at $x = l$. (10 points)

   (b). Show that there is no maximal principle for the wave equation. (10 points)

   (c). Let $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of

   $$\Delta u + \sum_{k=1}^{n} a_k(x)\frac{\partial u}{\partial x_k} + c(x)u = 0,$$ where $c(x) < 0$ in Ω.

   Show that $u = 0$ on $\partial\Omega$ implies $u = 0$ in Ω. (10 points)

3. Show that the modified Green's function for the boundrary value problem

$$-u'' = f\ ,\ \ 0 < x < 1\ ,\ \ u(0) = u(1)\ ,\ \ u'(0) = u'(1)\ ,$$

where $f \in L^2(\bar{\Omega})$, and satifies $\int_0^1 f(x)dx = 0$

is $g(x,\xi) = \frac{1}{12} + \frac{(x-\xi)^2}{2} - \frac{1}{2}|x-\xi|$. (15 points)

4. Suppose that L is strongly elliptic of order $2m$ on a bounded domain $\bar{\Omega}$ and satisfies $(-1)^m \operatorname{Re} \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geq C|\xi|^{2m}$ for all $\xi \in R^n$, $x \in \bar{\Omega}$, and that $L = L^*$.

(a). Show that there is an orthonormal basis $\{u_j\}$ for $H_0(\Omega)$ consisting of eigenfunctions for L such that $u_j \in C^\infty(\bar{\Omega})$ for all j and u_j satisfies boundrary conditions $\partial_\nu^i u_j = 0$ on $\partial\Omega$ for $i = 1, 2, \cdots m-1$. The eigenvalues are real and only accumulate only at $+\infty$. (15 points)

(b) Show that there is an orthonormal basis $\{u_j\}$ for $L^2(\Omega)$ consisting of eigenfunctions for the Laplacian such that $u_j \in C^\infty(\bar{\Omega})$ and $u_j = 0$ on $\partial\Omega$ for all j. The eigenvalues are all negative. (10 points)

國立台灣師範大學數學系
105 學年度下學期博士班資格考試題
科目：偏微分方程

You have to answer the problems 1~5. You may do any one problem of 6 or 7 as a bonus.

1. Consider the initial-boundary value problem for the backwards heat equation in one spatial dimension:

$$\partial_t u = -\partial_x^2 u, \quad (t,x) \in [0,1]\times[0,1]. \qquad (1)$$

(a) Find all solutions to the equation (1) that satisfy the boundary condition $u(t,0)=u(t,1)=0$ $t\in[0,1]$ and the initial condition $u(0,x)=f(x)$, where $f(x)$ be a ***smooth*** function (i.e., it is infinitely differentiable) on $[0,1]$. (15 points)

(b) If $\max_{x\in[0,1]}|f(x)|\le\varepsilon$, where ε is a very small positive number, explain what conclusions can be reached about the "size" of the solution at $t=1$. The term "size" is defined here to be $\max_{x\in[0,1]}|u(t,x)|$. (8 points)

(c) Does this initial-boundary value problem well-posed? Explain your viewpoint. (7 points)

2. Suppose that $u\in C^\infty(R^3)$ be a harmonic function on R^3:

$$\Delta u(x)=0, \quad x\in R^3.$$

Assume that $|u(x)|\le\sqrt{\|x\|}$ for all x, where $\|\cdot\|$ be the Euclidean norm on R^3.

Show that $u(x)=0$ for all $x\in R^3$. (15 points)

3. Solve following initial value problem:

$$u_{xx}-3u_{tx}-4u_{tt}=0,$$
$$u(0,x)=x^2, \quad u_t(0,x)=e^x. \text{ (15 points)}$$

4. Let $u(t,x)\in C^{1,2}([0,2]\times[0,1])$ be a solution to the following initial-boundary value problem:

$$\partial_t u-\partial_x^2 u=-u, \quad (t,x)\in[0,2]\times[0,1],$$
$$u(0,x)=f(x), \quad x\in[0,1],$$
$$u(t,0)=g(t), \quad u(t,1)=h(t), \quad t\in[0,2].$$

Assume that $f(x)\le 0$ for $x\in[0,1]$ and $g(t)\le 0$, $h(t)\le 0$ for $t\in[0,2]$. Prove that $u(t,x)\le 0$ holds for all $(t,x)\in[0,2]\times[0,1]$. (15 points)

5. Let $u(t,x)\in C^{1,2}([0,2]\times[0,1])$ be a solution to the following initial-boundary value problem:

$$\partial_t u-\partial_x^2 u=-u, \quad (t,x)\in[0,\infty)\times[0,1],$$
$$u(0,x)=f(x), \quad x\in[0,1],$$
$$u_x(t,0)=0, \quad u_x(t,1)=0, \quad t\in[0,\infty).$$

Define

$$T(t)=\int_0^1 u(t,x)dx.$$

(a) Show that $T(t)$ is constant in time (i.e., $T(t)=T(0)$ for all $t\ge 0$). (12 points)

(b) What happens to $u(t,x)$ as $t\to\infty$? Prove your guess. (13 points)

6. Assume that $h(t,x) \in C^2([0,\infty)\times R)$, that $f(x) \in C^2(R) \cap L^2(R)$, and that $g(x) \in C^1(R) \cap L^2(R)$. Let $u(t,x) \in C^2([0,\infty)\times R)$ be the solution to the following global Cauchy problem for an inhomogeneous wave equation:

$$-\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = h(t,x), \quad (t,x) \in [0,\infty)\times R,$$
$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x).$$

Assume that at each fixed t,

$$\| h(t,\cdot) \|_{L^2} \le \frac{1}{1+t^2}.$$

Also assume that at each fixed t, there exists a positive number $R(t)$ such that $u(t,x) = 0$ whenever $|x| \ge R(t)$. Define

$$E^2(t) = \int_R ((\partial_t u(t,x))^2 + (\partial_x u(t,x))^2) dx.$$

(a) Show that

$$\frac{d}{dt} E^2(t) = -2\int_R h(t,x)\partial_t u(t,x) dx. \text{ (10 points)}$$

(b) Show that $E(t) \le E(0) + C$ for all $t \ge 0$, where $C > 0$ is a constant. (10 points)

7. Let $f: R^n \to R$ be a smooth compactly supported function. Let $u(t,x)$ be the unique smooth solution to the following global Caucy problem:

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \quad (t,x) \in [0,\infty)\times R^n,$$
$$u(0,x) = f(x), \quad x \in R^n,$$
$$\partial_t u(0,x) = 0, \quad x \in R^n.$$

Let

$$\hat{u}(t,\xi) = \int_{R^n} e^{-2\pi i \xi\cdot x} f(x) d^n x$$

be the Fourier transform of $u(t,x)$ with respect to the spatial variable only.

(a) Show that $\hat{u}(t,\xi)$ is a solution to the following initial value problem:

$$\partial_t^2 \hat{u}(t,\xi) = -4\pi^2 |\xi|^2 \hat{u}(t,x), \quad (t,\xi) \in [0,\infty)\times R^n,$$
$$\hat{u}(0,\xi) = \hat{f}(\xi), \quad \xi \in R^n,$$
$$\partial_t \hat{u}(0,\xi) = 0, \quad \xi \in R^n. \quad \text{(10 points)}$$

(b) Find an expression for the solution $\hat{u}(t,\xi)$ of above initial value problem in terms of $\hat{f}(\xi)$ (and some other functions of (t,ξ). (Hint: If done correctly and simplified, your answer should involve a trigonometric function.) (10 points)