# 國立台灣師範大學數學系
# 103 學年度下學期博士班資格考試題
# 科目：微分方程/PDEs

Time and Date: 9-12, April 22, 2015

1. (10 pt.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u = e^{x+2t}$$

with $u(x,0) = 0$. Find the explicit solution of this equation.

2. (10 pt.) Solve the wave equation of $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^{2x},\ \forall\ (x,t) \in \mathbb{R}^2,$$

with the conditions, $u(x,0) = 0 = u(0,t)$, $\forall\, t, x \in \mathbb{R}$.

3. (20 pt.) Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a real-valued function, and

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy.$$

Show with details that

**(a)**

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1,\ \forall\, t > 0.$$

**(b)** For any fixed $x \in \mathbb{R}^n$,

$$\lim_{t \to +0} u(x,t) = g(x).$$

4. (10 pt.) Let $\mathbb{R}^2_+$ be the set of upper half-plane. Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be a harmonic function satisfying $u(x,y) = f(x/y)$, and the boundary conditions $u(x,0) = 1$ for $x > 0$ and $u(x,0) = 0$ for $x < 0$. Find the explicit formula of the solution $u(x,y)$.

5. (30 pt.) Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth harmonic function, i.e. $\Delta u = 0$. Denote the gradient of $u$ by $\nabla u$, and the Hessian (matrix) of $u$ by $\nabla^2 u$.

   **(a)** Show that $|\nabla u|^2$ is subharmonic, i.e. $\Delta |\nabla u|^2 \geq 0$.

   **(b)** Show that for any $k \in [0,n]$,

$$\frac{d}{dr}\left(\frac{1}{r^k}\int_{\mathbb{B}_r(0)} |\nabla u|^2 \, dx\right) \geq 0$$

   where $\mathbb{B}_r(0) := \{x \in \mathbb{R}^n : |x| < r\}$.

   **(c)** Let $n = 2$, and assume that $\det \nabla^2 u \neq 0$ at a point $p \in \mathbb{R}^2$. Show that $\det \nabla^2 u$ is superharmonic (i.e. $\Delta \det \nabla^2 u \leq 0$) in a neighborhood of $p$.

6. (20 pt.) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded simply-connected subset.

   **(a)** Give the definition of Sobolev spaces $W^{1,2}(\Omega)$ is in terms of the notion of weak derivatives. Is $W^{1,2}(\Omega)$ a Hilbert space (explain your answer)?

   **(b)** Let $p \in [1,n)$. If we want to establish an estimate of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

   for any function $u \in C_c^\infty(\mathbb{R}^n)$ and certain constants $C > 0$, $q \in [1,\infty)$, what should the algebraic relation of $p$, $q$, and $n$ be?

   (Hint: scaling of $u$ in either the domaun or the range would provide the information)

# 國立台灣師範大學數學系
# 104 學年度上學期博士班資格考試題
# 科目：偏微分方程
# Math/NTNU Qualifying Exam of PDEs in Oct. 2015

Time and Date: 2-5 PM, October 31, 2015

1. (15 pt.)

   Let $u : [0,\pi] \times (\mathbb{R}_+ \cup \{0\}) \to \mathbb{R}$ fulfill

   $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,\ \forall\ (x,t) \in (0,\pi) \times \mathbb{R}_+$$

   with the initial data

   $$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin nx, \quad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \beta_n \sin nx,$$

   and boundary conditions

   $$u(0,t) = 0 = u(\pi,t), \quad \forall\ t > 0.$$

   Represent the solution $u$ as a Fourier series

   $$u(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) \sin nx,$$

   and compute the coefficients $\gamma_n(t)$.

2. (25 pt.)

   Let

   $$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4\ t}}, \forall\ (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

Assume that there exist constant numbers, $M > 0$ and $\alpha \in (0,1)$, such that the real-valued function $f \in C(\mathbb{R}\times\mathbb{R}) \cap L^\infty(\mathbb{R}\times\mathbb{R})$ fulfills

$$|f(x_2,t_2) - f(x_1,t_1)| \le M \cdot (|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2})$$

for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R}^2$. Let

$$z(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-y,t-\tau) \cdot f(y,\tau)\, dy\, d\tau.$$

Show with details that

**(a)** (20 pt.) $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x^2}$ are continuous in $\mathbb{R}\times(\mathbb{R}_+ \cup \{0\})$.

**(b)** (5 pt.) The equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f$$

holds in the domain $\mathbb{R}\times\mathbb{R}_+$.

**Hint:** Consider the family of functions,

$$z_h(x,t) = \int_0^{t-h} \int_{-\infty}^{\infty} K(x-y,t-\tau) \cdot f(y,\tau)\, dy\, d\tau,$$

where $h \in (0,t/2)$.

3. (30 pt.) The following is a standard procedure to establish regularity of weak solutions of elliptic PDEs.

   Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, and simply-connected subset. Assume that $\Gamma : \mathbb{R} \to \mathbb{R}$ is smooth and bounded (i.e., $|\Gamma| \le M$ for some constant $M > 0$). Suppose that $v : \Omega \to \mathbb{R}$ is a weak solution of

   $$\Delta u + \Gamma(u)|\nabla u|^2 = 0 \tag{1}$$

   in the Sobolev space $W^{1,2}(\Omega)$. If $v$ is a weak solution of Eq.(1) in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$, where $p > d$, then the $L^p$-theory of elliptic PDEs implies

   $$v \in W^{2,q}(\Omega')$$

   for some $q \in (1,\infty)$ and any proper open set $\Omega' \subset \Omega$. The so-called bootstrapping argument is to proceed this procedure until one derives the interior smoothness of $v$, i.e. $v \in C^\infty(\Omega)$.

**(a)** (10 pt.) Give the definitions of weak derivatives and weak solutions of Eq.(1) in $W^{1,2}(\Omega)$.

**(b)** (20 pt.) Explain how to apply the boot-strapping argument to derive interior smoothness of $v$, i.e. prove that $v \in C^\infty(\Omega)$.

**Hints: You should first figure out `$q = ?$' in each step stated above. In other words, in the $L^p$-theory, $\Delta v = f \in L^r$ for some $r > 1$ implies that $v \in W^{2,s}(\Omega')$, where $s = ?$**

4. (30 pt.) Denote by $\mathbb{B}_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^d$. Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$u(x) = \frac{x}{|x|}.$$

**(a)** As $d = 1$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1, \infty)$? If it is yes, what is the range of $p$? If it is not, explain why.

**(b)** As $d = 2$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1, \infty)$? If it is yes, what is the range of $p$? If it is not, explain why.

**(c)** As $d = 3$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1, \infty)$? If it is yes, what is the range of $p$? If it is not, explain why.

# 國立台灣師範大學數學系
# 104 學年度下學期博士班資格考試題
# 科目：偏微分方程
# Math/NTNU Qualify Exam of PDEs on April 30, 2016

Time and Date: 3 hours, April 30, 2016

總共 5 大題，滿分 110 分。

1. (20 pt.) Denote by $\mathbb{U}_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^d$. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$u(x) = \frac{x_1}{\sqrt{\sum_{j=1}^{d} x_j^2}}.$$

**(a)** As $d = 1$, is it true that $u \in W^{1,p}(\mathbb{U}_1)$ for some $p \in [1,\infty)$? If the answer is positive, what is the range of $p$? If it is negative, explain why.

**(b)** As $d = 2$, is it true that $u \in W^{1,p}(\mathbb{U}_1)$ for some $p \in [1,\infty)$? If the answer is positive, what is the range of $p$? If it is negative, explain why.

2. (20 pt.) Denote by $C_c^\infty(\mathbb{R}^d)$ the class of smooth real-valued functions with compact support in $\mathbb{R}^d$.

**(a)** Show that any function $u \in C_c^\infty(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} (\Delta u)^2 \, dx = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 dx, \tag{1}$$

where $\Delta$ denotes the Laplace operator in $\mathbb{R}^d$.

**(b)** Explain why Eq.(1) also holds for any function $u \in C_c^2(\mathbb{R}^d)$.

3. (10 pt.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u = e^{2x-t}$$

with $u(x,0) = 0$. Find the explicit solution of this equation.

4. (30 pt.) Denote by $\mathbb{U}_R := \{x \in \mathbb{R}^2 : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^2$, and by $\mathbb{B}_R := \{x \in \mathbb{R}^2 : |x| \leq R\}$ the closed ball of radius $R > 0$ with center at the origin of $\mathbb{R}^2$. Suppose the functions $g$ and $u : \mathbb{B}_R \to \mathbb{R}$ are continuous and $u$ satisfies the Poisson equation,

$$\Delta u = 0, \text{ in } \mathbb{U}_R,$$

with boundary value $g$, i.e.

$$\lim_{x \to x_0} u(x) = g(x_0), \ \forall\, x_0 \in \partial \mathbb{B}_R.$$

Answer the following questions with sufficient details.

**(a)** The fundamental solution of Laplace equation is given by

$$\Gamma(x,y) = \frac{1}{2\pi} \log |x-y|,$$

where $x, y \in \mathbb{R}^2$. Show that the Poisson representation formula is given by

$$u(x) = \frac{R^2 - |x|^2}{2\pi R} \int_{y \in \partial \mathbb{B}_R} \frac{g(y)}{|x-y|^2} \, do(y), \ \forall\, x \in \mathbb{U}_R,$$

where $do(y)$ represents the arclength element of $\partial \mathbb{B}_R$ at $y$. (Hint: you might need Schwartz reflection principle to construct the so-called Green's functions and apply Green's identity.)

**(b)** Show that

$$\lim_{x \to x_0} u(x) = g(x_0),$$

for any $x_0 \in \partial \mathbb{B}_R$.

**(c)** There are several methods to prove Maximum Principle for harmonic functions. Could you just use the Poisson representation formula to prove the strong Maximum Principle of the harmonic function $u$? Namely, if

$$\sup_{\mathbb{B}_R} u = u(p), \text{ for some } p \in \mathbb{U}_R,$$

then $u$ is a constant function.

5. (30 pt.) Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbb{R}$ and $t > 0$. Let $g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a real-valued function, and

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) g(y)\, dy.$$

Show with details that

**(a)** the function $u$ satisfies the heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

in $\mathbb{R} \times \mathbb{R}_+$.

**(b)** For any fixed $x \in \mathbb{R}$,

$$\lim_{t \to +0} u(x,t) = g(x).$$

**(c)** if

$$\int_{\mathbb{R}} |g(x)|^2\, dx \leq M,$$

for some constant $M > 0$, then there exists a constant $C$ such that

$$|u(x,t)| \leq \frac{C}{t^{1/4}},$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$,

# PDE Qualify Exam

2016/10/31

1. Solve following problems. (10 points for each problem)

(1). $$\begin{cases} \dfrac{1}{(1+x)^2}\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}=0 \text{ for } 0<x<1,\ t>0; \\ u(0,t)=0; \\ u(1,t)=0; \\ u(x,0)=0; \\ \dfrac{\partial u}{\partial t}(x,0)=g(x). \end{cases}$$

(2). $$\begin{cases} \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0 \text{ for } r<1; \\ u(1,\theta)=\sin^2\theta. \end{cases}$$

(3). $$\begin{cases} \Delta u-u=0 & \text{for } 0<x<\pi,\ 0<y<\pi/2,\ 0<z<1; \\ u=0 & \text{for } x=0,\ y=0,\ z=1; \\ \dfrac{\partial u}{\partial x}=0 & \text{for } x=\pi; \\ \dfrac{\partial u}{\partial x}=0 & \text{for } y=\dfrac{\pi}{2}; \\ \dfrac{\partial u}{\partial z}(x,y,0)=2x-\pi. \end{cases}$$

2. (a). Show that if $\begin{cases} \dfrac{\partial u}{\partial t}-k\dfrac{\partial^2 u}{\partial x^2}=0 \text{ for } 0<x<l; \\ \dfrac{\partial u}{\partial x}(0,t)=0. \end{cases}$ the maximum of $u$ for $0\le x\le l$ and $0\le t\le t_1$ must occur at $t=0$ or at $x=l$. (10 points)

(b). Show that there is no maximal principle for the wave equation. (10 points)

(c). Let $u(x)\in C^2(\Omega)\bigcap C(\bar{\Omega})$ be a solution of

$$\Delta u+\sum_{k=1}^{n}a_k(x)\frac{\partial u}{\partial x_k}+c(x)u=0, \text{ where } c(x)<0 \text{ in } \Omega.$$

Show that $u=0$ on $\partial\Omega$ implies $u=0$ in $\Omega$. (10 points)

3. Show that the modified Green's function for the boundrary value problem

$$-u'' = f\ ,\quad 0 < x < 1\ ,\quad u(0) = u(1)\ ,\quad u'(0) = u'(1)\ ,$$

where $f \in L^2(\bar{\Omega})$, and satifies $\int_0^1 f(x)dx = 0$

is $g(x,\xi) = \frac{1}{12} + \frac{(x-\xi)^2}{2} - \frac{1}{2}|\,x-\xi\,|$. (15 points)

4. Suppose that $L$ is strongly elliptic of order $2m$ on a bounded domain $\bar{\Omega}$ and satisfies $(-1)^m \operatorname{Re} \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \ge C|\,\xi\,|^{2m}$ for all $\xi \in R^n$, $x \in \bar{\Omega}$, and that $L = L^*$.

(a). Show that there is an orthonormal basis $\{u_j\}$ for $H_0(\Omega)$ consisting of eigenfunctions for $L$ such that $u_j \in C^\infty(\bar{\Omega})$ for all $j$ and $u_j$ satisfies boundrary conditions $\partial_\nu^i u_j = 0$ on $\partial\Omega$ for $i = 1, 2, \cdots m-1$. The eigenvalues are real and only accumulate only at $+\infty$. (15 points)

(b) Show that there is an orthonormal basis $\{u_j\}$ for $L^2(\Omega)$ consisting of eigenfunctions for the Laplacian such that $u_j \in C^\infty(\bar{\Omega})$ and $u_j = 0$ on $\partial\Omega$ for all $j$. The eigenvalues are all negative. (10 points)

# 國立台灣師範大學數學系
# 105 學年度下學期博士班資格考試題
# 科目：偏微分方程

**You have to answer the problems 1~5. You may do any one problem of 6 or 7 as a bonus.**

1. Consider the initial-boundary value problem for the backwards heat equation in one spatial dimension:
$$\partial_t u = -\partial_x^2 u, \quad (t,x) \in [0,1]\times[0,1]. \tag{1}$$
   (a) Find all solutions to the equation (1) that satisfy the boundary condition $u(t,0)=u(t,1)=0$ $t\in[0,1]$ and the initial condition $u(0,x)=f(x)$, where $f(x)$ be a ***smooth*** function (i.e., it is infinitely differentiable) on $[0,1]$. (15 points)
   (b) If $\max_{x\in[0,1]}|f(x)|\le\varepsilon$, where $\varepsilon$ is a very small positive number, explain what conclusions can be reached about the "size" of the solution at $t=1$. The term "size" is defined here to be $\max_{x\in[0,1]}|u(t,x)|$. (8 points)
   (c) Does this initial-boundary value problem well-posed? Explain your viewpoint. (7 points)

2. Suppose that $u\in C^\infty(R^3)$ be a harmonic function on $R^3$:
$$\Delta u(x)=0, \quad x\in R^3.$$
   Assume that $|u(x)|\le\sqrt{\|x\|}$ for all $x$, where $\|\cdot\|$ be the Euclidean norm on $R^3$.
   Show that $u(x)=0$ for all $x\in R^3$. (15 points)

3. Solve following initial value problem:
$$u_{xx}-3u_{tx}-4u_{tt}=0,$$
$$u(0,x)=x^2, \quad u_t(0,x)=e^x. \text{ (15 points)}$$

4. Let $u(t,x)\in C^{1,2}([0,2]\times[0,1])$ be a solution to the following initial-boundary value problem:
$$\partial_t u-\partial_x^2 u=-u, \quad (t,x)\in[0,2]\times[0,1],$$
$$u(0,x)=f(x), \quad x\in[0,1],$$
$$u(t,0)=g(t), \quad u(t,1)=h(t), \quad t\in[0,2].$$
   Assume that $f(x)\le 0$ for $x\in[0,1]$ and $g(t)\le 0$, $h(t)\le 0$ for $t\in[0,2]$. Prove that $u(t,x)\le 0$ holds for all $(t,x)\in[0,2]\times[0,1]$. (15 points)

5. Let $u(t,x)\in C^{1,2}([0,2]\times[0,1])$ be a solution to the following initial-boundary value problem:
$$\partial_t u-\partial_x^2 u=-u, \quad (t,x)\in[0,\infty)\times[0,1],$$
$$u(0,x)=f(x), \quad x\in[0,1],$$
$$u_x(t,0)=0, \quad u_x(t,1)=0, \quad t\in[0,\infty).$$
   Define
$$T(t)=\int_0^1 u(t,x)\,dx.$$
   (a) Show that $T(t)$ is constant in time (i.e., $T(t)=T(0)$ for all $t\ge 0$). (12 points)
   (b) What happens to $u(t,x)$ as $t\to\infty$? Prove your guess. (13 points)

6. Assume that $h(t,x) \in C^2([0,\infty)\times R)$, that $f(x) \in C^2(R) \cap L^2(R)$, and that $g(x) \in C^1(R) \cap L^2(R)$. Let $u(t,x) \in C^2([0,\infty)\times R)$ be the solution to the following global Cauchy problem for an inhomogeneous wave equation:

$$-\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = h(t,x), \quad (t,x) \in [0,\infty)\times R,$$
$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x).$$

Assume that at each fixed $t$,

$$\| h(t,\cdot) \|_{L^2} \le \frac{1}{1+t^2}.$$

Also assume that at each fixed $t$, there exists a positive number $R(t)$ such that $u(t,x) = 0$ whenever $|x| \ge R(t)$. Define

$$E^2(t) = \int_R ((\partial_t u(t,x))^2 + (\partial_x u(t,x))^2) dx.$$

(a) Show that

$$\frac{d}{dt} E^2(t) = -2\int_R h(t,x)\partial_t u(t,x) dx. \text{ (10 points)}$$

(b) Show that $E(t) \le E(0) + C$ for all $t \ge 0$, where $C > 0$ is a constant. (10 points)

7. Let $f : R^n \to R$ be a smooth compactly supported function. Let $u(t,x)$ be the unique smooth solution to the following global Caucy problem:

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \quad (t,x) \in [0,\infty)\times R^n,$$
$$u(0,x) = f(x), \quad x \in R^n,$$
$$\partial_t u(0,x) = 0, \quad x \in R^n.$$

Let

$$\hat{u}(t,\xi) = \int_{R^n} e^{-2\pi i \xi \cdot x} f(x) d^n x$$

be the Fourier transform of $u(t,x)$ with respect to the spatial variable only.

(a) Show that $\hat{u}(t,\xi)$ is a solution to the following initial value problem:

$$\partial_t^2 \hat{u}(t,\xi) = -4\pi^2 |\xi|^2 \hat{u}(t,x), \quad (t,\xi) \in [0,\infty)\times R^n,$$
$$\hat{u}(0,\xi) = \hat{f}(\xi), \quad \xi \in R^n,$$
$$\partial_t \hat{u}(0,\xi) = 0, \quad \xi \in R^n. \quad \text{(10 points)}$$

(b) Find an expression for the solution $\hat{u}(t,\xi)$ of above initial value problem in terms of $\hat{f}(\xi)$ (and some other functions of $(t,\xi)$. (Hint: If done correctly and simplified, your answer should involve a trigonometric function.) (10 points)

**109 學年度上學期博士班資格考試題**
**科目: 偏微分方程**
2020 年 10 月 30 日

1. Solve the following initial boundary value problem

$$\begin{array}{l} u_t = u_{xx} + 5, \quad 0 < x < \pi, \quad t > 0 \\ u(0,t) = 1, \quad u(\pi,t) = 6, \quad t > 0 \\ u(x,0) = 1 + \frac{5}{\pi}x + 2\sin 3x, \quad 0 < x < \pi. \end{array}$$

2. (a) State any version of maximum principle for heat equation in a bounded domain.

   (b) Let $\Omega$ denote an open bounded set of $\mathbf{R}^n$ and $T > 0$ be a fix number. Prove a uniqueness theorem for the following initial boundary value problem

$$\begin{array}{lll} u_t - \Delta u = f, & \text{in} & \Omega \times (0,T) \\ u(x,0) = g(x), & \text{in} & \Omega \\ u = 0, & \text{on} & \partial\Omega \times (0,T) \end{array}$$

   where $f$ and $g$ are continuous such that $g = 0$ on $\partial\Omega$.

3. Let $\Omega$ be a a region in $\mathbf{R}^n$ and $u \in C^2(\Omega)$. Show that $\triangle u \geq 0$ in $\Omega$ if and only if for each $\xi \in \Omega$ :

$$u(\xi) \leq \frac{1}{\omega_n \rho^{n-1}} \int_{|x-\xi|=\rho} u(x)\, dS_x$$

   for all $\rho$ sufficiently small, where $\omega_n$ is the surface area of the unit sphere in $\mathbf{R}^n$.

4. (a) Define the notion of distribution.

   (b) Let $u$ be a distribution on $\mathbf{R}$ and suppose that $u' = 0$ on $\mathbf{R}$.
   Show that $u =$ constant; i.e. show that there is a number $a$ such that

$$u(\phi) = \int_{\mathbf{R}} a\phi\, dx \text{ for all } \phi \in C_0^\infty(\mathbf{R}).$$

5. (a) Let $u \in W_0^{1,2}$ satisfy

$$\int_\Omega \nabla u \cdot \nabla \phi\, dx \geq 0 \quad \forall \phi \in W_0^{1,2}, \quad \phi \geq 0.$$

   Show that $u \geq 0$ a.e. in $\Omega$.

   (b) Let $u \in W_0^{1,2}$ satisfy the inequality in (a), show that

$$\inf_\Omega u \geq \inf_{\partial\Omega} u \quad \text{(essinf)}$$

# 109 學年度下學期博士班資格考試題
# 科目: 偏微分方程

2021 年 04 月 26 日

1. Solve the following initial boundary value problem

$$\begin{aligned} &4u_t = u_{xx}, && 0 < x < 1, \quad t > 0, \\ &u(x,0) = a_1 \sin(\pi x) + \cdots + a_m \sin(m\pi x), && 0 \le x \le 1, \\ &u(0,t) = u(1,t) = 0, && t > 0 \end{aligned}$$

where $a_1, \cdots, a_m$ are constants and $m$ is a positive integer.

2. State and prove the mean value property for harmonic functions.

3. (a) Let $\Omega$ be a bounded domain and $\Gamma$ be a nonempty open subset of $\partial\Omega$ such that $\Gamma$ is real analysis. Suppose that $\Delta u = 0$ in $\Omega$, $u = 0, \nabla u = 0$ on $\Gamma$. Show that $u$ is identically equal to zero in $\Omega$.

   (b) State (without proof) carefully the theorems you used in (a)

4. Let $K$ be a compact subset in $\mathbf{R}^n$. Define $f$ to be uniform Hölder continuous with exponent $\alpha \in (0,1]$ in $K$, (denoted by $f \in C^\alpha(K)$), if

$$\sup_{x,y\in K, x\neq y} \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \right\} < \infty.$$

Show that $fg \in C^\gamma(K)$ if $f \in C^\alpha(K)$ and $g \in C^\beta(K)$, where $\gamma = \min(\alpha, \beta)$.

5. Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbf{R}^n$ and $t > 0$.

   (a) Show that

$$\int_{\mathbf{R}^n} \Phi(x,t)\, dx = 1, \quad \forall t > 0.$$

   (b) Let $f \in C(\mathbf{R}^n) \bigcap L^\infty(\mathbf{R}^n)$ be a real-valued function, and

$$u(x,t) = \int_{\mathbf{R}^n} \Phi(x-y,t) f(y)\, dy.$$

   Show that, for any fixed $x \in \mathbf{R}^n$,

$$\lim_{t\to 0^+} u(x,t) = f(x).$$