PhD Qualify Exam in Numerical Analysis

Spring 2024

(1) Consider the special model problem

$$\begin{cases} -u''(x) + u(x) = (1+\pi^2)\sin \pi x, \ \text{for } 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

which exact solution is $u(x) = \sin(\pi x)$.

(a) (10%) Please use the second order centered difference method with uniform mesh to discretize (1) and write down the associated linear system

$$A\mathbf{u}_h = \mathbf{b}, \quad A \in \mathbb{R}^{n\times n}, \tag{2}$$

where n is the mesh number.

(b) (15%) Show that the coefficient matrix A in (2) is not only strictly diagonal dominant but also symmetric positive definite.

(c) (15%) Prove that for any initial vector, both the Jacobi and Gauss-Seidel methods converge to the unique solution of (2). Which one is superior? Why?

(d) (10%) Compare the conjugate gradient method with the gradient method in solving linear system (2).

(e) (10%) From a computational efficiency perspective, which method would you choose to solve this linear system? Why?

(2) (10%) Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.

(3) (15%) Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ that uses $f(x_0-h)$, $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$, and $f(x_0+3h)$.

(4) (15%) Consider the nonlinear system

$$x_1^2 - 10x_1 + x_2^2 + 8 = 0, \quad x_1 x_2^2 + x_1 - 10x_2 + 8 = 0 \ \text{ for } 0 \le x_1, x_2 \le 1.5. \tag{3}$$

Please derive an iterative method to solve (3) and prove the convergence.