國立台灣師範大學數學系
103 學年度下學期博士班資格考試題
科目：微分方程/PDEs

Time and Date: 9-12, April 22, 2015

1. (10 pt.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u = e^{x+2t}$$

with $u(x,0) = 0$. Find the explicit solution of this equation.

2. (10 pt.) Solve the wave equation of $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^{2x},\ \forall\ (x,t) \in \mathbb{R}^2,$$

with the conditions, $u(x,0) = 0 = u(0,t)$, $\forall\, t,x \in \mathbb{R}$.

3. (20 pt.) Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a real-valued function, and

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy.$$

Show with details that

(a)

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1,\ \forall\, t > 0.$$

(b) For any fixed $x \in \mathbb{R}^n$,

$$\lim_{t \to +0} u(x,t) = g(x).$$

4. (10 pt.) Let $\mathbb{R}^2_+$ be the set of upper half-plane. Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be a harmonic function satisfying $u(x,y) = f(x/y)$, and the boundary conditions $u(x,0) = 1$ for $x > 0$ and $u(x,0) = 0$ for $x < 0$. Find the explicit formula of the solution $u(x,y)$.

5. (30 pt.) Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth harmonic function, i.e. $\Delta u = 0$. Denote the gradient of u by ∇u, and the Hessian (matrix) of u by $\nabla^2 u$.

   (a) Show that $|\nabla u|^2$ is subharmonic, i.e. $\Delta |\nabla u|^2 \geq 0$.

   (b) Show that for any $k \in [0, n]$,

$$\frac{d}{dr}\left(\frac{1}{r^k}\int_{\mathbb{B}_r(0)} |\nabla u|^2 \, dx\right) \geq 0$$

   where $\mathbb{B}_r(0) := \{x \in \mathbb{R}^n : |x| < r\}$.

   (c) Let $n = 2$, and assume that $\det \nabla^2 u \neq 0$ at a point $p \in \mathbb{R}^2$. Show that $\det \nabla^2 u$ is superharmonic (i.e. $\Delta \det \nabla^2 u \leq 0$) in a neighborhood of p.

6. (20 pt.) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded simply-connected subset.

   (a) Give the definition of Sobolev spaces $W^{1,2}(\Omega)$ is in terms of the notion of weak derivatives. Is $W^{1,2}(\Omega)$ a Hilbert space (explain your answer)?

   (b) Let $p \in [1, n)$. If we want to establish an estimate of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

   for any function $u \in C_c^\infty(\mathbb{R}^n)$ and certain constants $C > 0$, $q \in [1, \infty)$, what should the algebraic relation of p, q, and n be?

   (Hint: scaling of u in either the domaun or the range would provide the information)

國立台灣師範大學數學系
104 學年度上學期博士班資格考試題
科目：偏微分方程
Math/NTNU Qualifying Exam of PDEs in Oct. 2015

Time and Date: 2-5 PM, October 31, 2015

1. (15 pt.)

   Let $u : [0,\pi] \times (\mathbb{R}_+ \cup \{0\}) \to \mathbb{R}$ fulfill

   $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,\ \forall\ (x,t) \in (0,\pi) \times \mathbb{R}_+$$

   with the initial data

   $$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin nx, \quad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \beta_n \sin nx,$$

   and boundary conditions

   $$u(0,t) = 0 = u(\pi,t), \quad \forall\ t > 0.$$

   Represent the solution u as a Fourier series

   $$u(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) \sin nx,$$

   and compute the coefficients $\gamma_n(t)$.

2. (25 pt.)

   Let

   $$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4\ t}},\ \forall\ (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

Assume that there exist constant numbers, $M > 0$ and $\alpha \in (0,1)$, such that the real-valued function $f \in C(\mathbb{R}\times\mathbb{R}) \cap L^\infty(\mathbb{R}\times\mathbb{R})$ fulfills

$$|f(x_2,t_2) - f(x_1,t_1)| \leq M \cdot (|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2})$$

for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R}^2$. Let

$$z(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-y,t-\tau)\cdot f(y,\tau)\, dy\, d\tau.$$

Show with details that

(a) (20 pt.) $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x^2}$ are continuous in $\mathbb{R}\times(\mathbb{R}_+ \cup \{0\})$.

(b) (5 pt.) The equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f$$

holds in the domain $\mathbb{R}\times\mathbb{R}_+$.

Hint: Consider the family of functions,

$$z_h(x,t) = \int_0^{t-h} \int_{-\infty}^{\infty} K(x-y,t-\tau)\cdot f(y,\tau)\, dy\, d\tau,$$

where $h \in (0,t/2)$.

3. (30 pt.) The following is a standard procedure to establish regularity of weak solutions of elliptic PDEs.

   Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, and simply-connected subset. Assume that $\Gamma : \mathbb{R} \to \mathbb{R}$ is smooth and bounded (i.e., $|\Gamma| \leq M$ for some constant $M > 0$). Suppose that $v : \Omega \to \mathbb{R}$ is a weak solution of

$$\Delta u + \Gamma(u)|\nabla u|^2 = 0 \tag{1}$$

   in the Sobolev space $W^{1,2}(\Omega)$. If v is a weak solution of Eq.(1) in $W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$, where $p > d$, then the L^p-theory of elliptic PDEs implies

$$v \in W^{2,q}(\Omega')$$

   for some $q \in (1,\infty)$ and any proper open set $\Omega' \subset \Omega$. The so-called bootstrapping argument is to proceed this procedure until one derives the interior smoothness of v, i.e. $v \in C^\infty(\Omega)$.

(a) (10 pt.) Give the definitions of weak derivatives and weak solutions of Eq.(1) in $W^{1,2}(\Omega)$.

(b) (20 pt.) Explain how to apply the boot-strapping argument to derive interior smoothness of v, i.e. prove that $v \in C^\infty(\Omega)$.

Hints: You should first figure out `$q =?$' in each step stated above. In other words, in the L^p-theory, $\Delta v = f \in L^r$ for some $r > 1$ implies that $v \in W^{2,s}(\Omega')$, where $s =?$

4. (30 pt.) Denote by $\mathbb{B}_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^d$. Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$u(x) = \frac{x}{|x|}.$$

(a) As $d = 1$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1,\infty)$? If it is yes, what is the range of p? If it is not, explain why.

(b) As $d = 2$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1,\infty)$? If it is yes, what is the range of p? If it is not, explain why.

(c) As $d = 3$, is it true that $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1,\infty)$? If it is yes, what is the range of p? If it is not, explain why.

國立台灣師範大學數學系
104 學年度下學期博士班資格考試題
科目：偏微分方程
Math/NTNU Qualify Exam of PDEs on April 30, 2016

Time and Date: 3 hours, April 30, 2016

總共 5 大題，滿分 110 分。

1. (20 pt.) Denote by $\mathbb{U}_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^d$. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$u(x) = \frac{x_1}{\sqrt{\sum_{j=1}^d x_j^2}}.$$

   (a) As $d = 1$, is it true that $u \in W^{1,p}(\mathbb{U}_1)$ for some $p \in [1, \infty)$? If the answer is positive, what is the range of p? If it is negative, explain why.

   (b) As $d = 2$, is it true that $u \in W^{1,p}(\mathbb{U}_1)$ for some $p \in [1, \infty)$? If the answer is positive, what is the range of p? If it is negative, explain why.

2. (20 pt.) Denote by $C_c^\infty(\mathbb{R}^d)$ the class of smooth real-valued functions with compact support in $\mathbb{R}^d$.

   (a) Show that any function $u \in C_c^\infty(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} (\Delta u)^2 \, dx = \sum_{i,j=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 dx, \qquad (1)$$

   where Δ denotes the Laplace operator in $\mathbb{R}^d$.

   (b) Explain why Eq.(1) also holds for any function $u \in C_c^2(\mathbb{R}^d)$.

3. (10 pt.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u = e^{2x-t}$$

with $u(x,0) = 0$. Find the explicit solution of this equation.

4. (30 pt.) Denote by $\mathbb{U}_R := \{x \in \mathbb{R}^2 : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^2$, and by $\mathbb{B}_R := \{x \in \mathbb{R}^2 : |x| \leq R\}$ the closed ball of radius $R > 0$ with center at the origin of $\mathbb{R}^2$. Suppose the functions g and $u : \mathbb{B}_R \to \mathbb{R}$ are continuous and u satisfies the Poisson equation,

$$\Delta u = 0, \text{ in } \mathbb{U}_R,$$

with boundary value g, i.e.

$$\lim_{x \to x_0} u(x) = g(x_0), \ \forall \, x_0 \in \partial \mathbb{B}_R.$$

Answer the following questions with sufficient details.

(a) The fundamental solution of Laplace equation is given by

$$\Gamma(x,y) = \frac{1}{2\pi} \log |x-y|,$$

where $x, y \in \mathbb{R}^2$. Show that the Poisson representation formula is given by

$$u(x) = \frac{R^2 - |x|^2}{2\pi R} \int_{y \in \partial \mathbb{B}_R} \frac{g(y)}{|x-y|^2} \, do(y), \ \forall \, x \in \mathbb{U}_R,$$

where $do(y)$ represents the arclength element of $\partial \mathbb{B}_R$ at y. (Hint: you might need Schwartz reflection principle to construct the so-called Green's functions and apply Green's identity.)

(b) Show that

$$\lim_{x \to x_0} u(x) = g(x_0),$$

for any $x_0 \in \partial \mathbb{B}_R$.

(c) There are several methods to prove Maximum Principle for harmonic functions. Could you just use the Poisson representation formula to prove the strong Maximum Principle of the harmonic function u? Namely, if

$$\sup_{\mathbb{B}_R} u = u(p), \text{ for some } p \in \mathbb{U}_R,$$

then u is a constant function.

5. (30 pt.) Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{1/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbb{R}$ and $t > 0$. Let $g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a real-valued function, and

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) g(y)\, dy.$$

Show with details that

(a) the function u satisfies the heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

in $\mathbb{R} \times \mathbb{R}_+$.

(b) For any fixed $x \in \mathbb{R}$,

$$\lim_{t \to +0} u(x,t) = g(x).$$

(c) if

$$\int_{\mathbb{R}} |g(x)|^2\, dx \le M,$$

for some constant $M > 0$, then there exists a constant C such that

$$|u(x,t)| \le \frac{C}{t^{1/4}},$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$,

PDE Qualify Exam

2016/10/31

1. Solve following problems. (10 points for each problem)

(1). $$\begin{cases} \dfrac{1}{(1+x)^2}\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}=0 \text{ for } 0<x<1,\ t>0; \\ u(0,t)=0; \\ u(1,t)=0; \\ u(x,0)=0; \\ \dfrac{\partial u}{\partial t}(x,0)=g(x). \end{cases}$$

(2). $$\begin{cases} \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0 \text{ for } r<1; \\ u(1,\theta)=\sin^2\theta. \end{cases}$$

(3). $$\begin{cases} \Delta u-u=0 \text{ for } 0<x<\pi,\ 0<y<\pi/2,\ 0<z<1; \\ u=0 \quad \text{for } x=0,\ y=0,\ z=1; \\ \dfrac{\partial u}{\partial x}=0 \quad \text{for } x=\pi; \\ \dfrac{\partial u}{\partial x}=0 \quad \text{for } y=\dfrac{\pi}{2}; \\ \dfrac{\partial u}{\partial z}(x,y,0)=2x-\pi. \end{cases}$$

2. (a). Show that if $\begin{cases} \dfrac{\partial u}{\partial t}-k\dfrac{\partial^2 u}{\partial x^2}=0 \text{ for } 0<x<l; \\ \dfrac{\partial u}{\partial x}(0,t)=0. \end{cases}$ the maximum of *u* for $0\le x\le l$ and $0\le t\le t_1$ must occur at $t=0$ or at $x=l$. (10 points)

   (b). Show that there is no maximal principle for the wave equation. (10 points)

   (c). Let $u(x)\in C^2(\Omega)\bigcap C(\bar{\Omega})$ be a solution of

   $$\Delta u+\sum_{k=1}^{n}a_k(x)\frac{\partial u}{\partial x_k}+c(x)u=0, \text{ where } c(x)<0 \text{ in } \Omega.$$

   Show that $u=0$ on $\partial\Omega$ implies $u=0$ in Ω. (10 points)

3. Show that the modified Green's function for the boundrary value problem

$$-u'' = f\ ,\ \ 0 < x < 1\ ,\ \ u(0) = u(1)\ ,\ \ u'(0) = u'(1)\ ,$$

where $f \in L^2(\bar{\Omega})$, and satifies $\int_0^1 f(x)dx = 0$

is $g(x,\xi) = \frac{1}{12} + \frac{(x-\xi)^2}{2} - \frac{1}{2}|x-\xi|$. (15 points)

4. Suppose that L is strongly elliptic of order $2m$ on a bounded domain $\bar{\Omega}$ and satisfies $(-1)^m \operatorname{Re} \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \ge C|\xi|^{2m}$ for all $\xi \in R^n$, $x \in \bar{\Omega}$, and that $L = L^*$.

(a). Show that there is an orthonormal basis $\{u_j\}$ for $H_0(\Omega)$ consisting of eigenfunctions for L such that $u_j \in C^\infty(\bar{\Omega})$ for all j and u_j satisfies boundrary conditions $\partial_\nu^i u_j = 0$ on $\partial\Omega$ for $i = 1, 2, \cdots m-1$. The eigenvalues are real and only accumulate only at $+\infty$. (15 points)

(b) Show that there is an orthonormal basis $\{u_j\}$ for $L^2(\Omega)$ consisting of eigenfunctions for the Laplacian such that $u_j \in C^\infty(\bar{\Omega})$ and $u_j = 0$ on $\partial\Omega$ for all j. The eigenvalues are all negative. (10 points)

國立台灣師範大學數學系
105 學年度下學期博士班資格考試題
科目：偏微分方程

You have to answer the problems 1~5. You may do any one problem of 6 or 7 as a bonus.

1. Consider the initial-boundary value problem for the backwards heat equation in one spatial dimension:

$$\partial_t u = -\partial_x^2 u, \quad (t,x) \in [0,1]\times[0,1]. \qquad (1)$$

(a) Find all solutions to the equation (1) that satisfy the boundary condition $u(t,0) = u(t,1) = 0$ $t \in [0,1]$ and the initial condition $u(0,x) = f(x)$, where $f(x)$ be a ***smooth*** function (i.e., it is infinitely differentiable) on $[0,1]$. (15 points)

(b) If $\max_{x\in[0,1]} |f(x)| \le \varepsilon$, where ε is a very small positive number, explain what conclusions can be reached about the "size" of the solution at $t = 1$. The term "size" is defined here to be $\max_{x\in[0,1]} |u(t,x)|$. (8 points)

(c) Does this initial-boundary value problem well-posed? Explain your viewpoint. (7 points)

2. Suppose that $u \in C^\infty(R^3)$ be a harmonic function on R^3:

$$\Delta u(x) = 0, \quad x \in R^3.$$

Assume that $|u(x)| \le \sqrt{\|x\|}$ for all x, where $\|\cdot\|$ be the Euclidean norm on R^3.

Show that $u(x) = 0$ for all $x \in R^3$. (15 points)

3. Solve following initial value problem:

$$u_{xx} - 3u_{tx} - 4u_{tt} = 0,$$
$$u(0,x) = x^2, \quad u_t(0,x) = e^x. \text{ (15 points)}$$

4. Let $u(t,x) \in C^{1,2}([0,2]\times[0,1])$ be a solution to the following initial-boundary value problem:

$$\partial_t u - \partial_x^2 u = -u, \quad (t,x) \in [0,2]\times[0,1],$$
$$u(0,x) = f(x), \quad x \in [0,1],$$
$$u(t,0) = g(t), \quad u(t,1) = h(t), \quad t \in [0,2].$$

Assume that $f(x) \le 0$ for $x \in [0,1]$ and $g(t) \le 0$, $h(t) \le 0$ for $t \in [0,2]$. Prove that $u(t,x) \le 0$ holds for all $(t,x) \in [0,2]\times[0,1]$. (15 points)

5. Let $u(t,x) \in C^{1,2}([0,2]\times[0,1])$ be a solution to the following initial-boundary value problem:

$$\partial_t u - \partial_x^2 u = -u, \quad (t,x) \in [0,\infty)\times[0,1],$$
$$u(0,x) = f(x), \quad x \in [0,1],$$
$$u_x(t,0) = 0, \quad u_x(t,1) = 0, \quad t \in [0,\infty).$$

Define

$$T(t) = \int_0^1 u(t,x)\,dx.$$

(a) Show that $T(t)$ is constant in time (i.e., $T(t) = T(0)$ for all $t \ge 0$). (12 points)

(b) What happens to $u(t,x)$ as $t \to \infty$? Prove your guess. (13 points)

6. Assume that $h(t,x) \in C^2([0,\infty)\times R)$, that $f(x) \in C^2(R)\bigcap L^2(R)$, and that $g(x) \in C^1(R)\bigcap L^2(R)$. Let $u(t,x) \in C^2([0,\infty)\times R)$ be the solution to the following global Cauchy problem for an inhomogeneous wave equation:

$$-\partial_t^2 u(t,x) + \partial_x^2 u(t,x) = h(t,x), \quad (t,x) \in [0,\infty)\times R,$$
$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x).$$

Assume that at each fixed t,

$$\| h(t,\cdot) \|_{L^2} \leq \frac{1}{1+t^2}.$$

Also assume that at each fixed t, there exists a positive number $R(t)$ such that $u(t,x) = 0$ whenever $|x| \geq R(t)$. Define

$$E^2(t) = \int_R ((\partial_t u(t,x))^2 + (\partial_x u(t,x))^2) dx.$$

(a) Show that

$$\frac{d}{dt} E^2(t) = -2\int_R h(t,x)\partial_t u(t,x) dx. \text{ (10 points)}$$

(b) Show that $E(t) \leq E(0) + C$ for all $t \geq 0$, where $C > 0$ is a constant. (10 points)

7. Let $f: R^n \to R$ be a smooth compactly supported function. Let $u(t,x)$ be the unique smooth solution to the following global Caucy problem:

$$-\partial_t^2 u(t,x) + \Delta u(t,x) = 0, \quad (t,x) \in [0,\infty)\times R^n,$$
$$u(0,x) = f(x), \quad x \in R^n,$$
$$\partial_t u(0,x) = 0, \quad x \in R^n.$$

Let

$$\hat{u}(t,\xi) = \int_{R^n} e^{-2\pi i \xi \cdot x} f(x) d^n x$$

be the Fourier transform of $u(t,x)$ with respect to the spatial variable only.

(a) Show that $\hat{u}(t,\xi)$ is a solution to the following initial value problem:

$$\partial_t^2 \hat{u}(t,\xi) = -4\pi^2 |\xi|^2 \hat{u}(t,x), \quad (t,\xi) \in [0,\infty)\times R^n,$$
$$\hat{u}(0,\xi) = \hat{f}(\xi), \quad \xi \in R^n,$$
$$\partial_t \hat{u}(0,\xi) = 0, \quad \xi \in R^n. \quad \text{(10 points)}$$

(b) Find an expression for the solution $\hat{u}(t,\xi)$ of above initial value problem in terms of $\hat{f}(\xi)$ (and some other functions of (t,ξ). (Hint: If done correctly and simplified, your answer should involve a trigonometric function.) (10 points)

109 學年度上學期博士班資格考試題
科目: 偏微分方程

2020 年 10 月 30 日

1. Solve the following initial boundary value problem

$$\begin{aligned} &u_t = u_{xx} + 5, \quad 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 1, \quad u(\pi,t) = 6, \quad t > 0 \\ &u(x,0) = 1 + \tfrac{5}{\pi}x + 2\sin 3x, \quad 0 < x < \pi. \end{aligned}$$

2. (a) State any version of maximum principle for heat equation in a bounded domain.

   (b) Let Ω denote an open bounded set of $\mathbf{R}^n$ and $T > 0$ be a fix number. Prove a uniqueness theorem for the following initial boundary value problem

$$\begin{aligned} &u_t - \Delta u = f, && \text{in} \quad \Omega \times (0,T) \\ &u(x,0) = g(x), && \text{in} \quad \Omega \\ &u = 0, && \text{on} \quad \partial\Omega \times (0,T) \end{aligned}$$

   where f and g are continuous such that $g = 0$ on $\partial\Omega$.

3. Let Ω be a a region in $\mathbf{R}^n$ and $u \in C^2(\Omega)$. Show that $\triangle u \geq 0$ in Ω if and only if for each $\xi \in \Omega$:

$$u(\xi) \leq \frac{1}{\omega_n \rho^{n-1}} \int_{|x-\xi|=\rho} u(x)\, dS_x$$

   for all ρ sufficiently small, where ω_n is the surface area of the unit sphere in $\mathbf{R}^n$.

4. (a) Define the notion of distribution.

   (b) Let u be a distribution on $\mathbf{R}$ and suppose that $u' = 0$ on $\mathbf{R}$.
   Show that $u =$ constant; i.e. show that there is a number a such that

$$u(\phi) = \int_{\mathbf{R}} a\phi\, dx \text{ for all } \phi \in C_0^\infty(\mathbf{R}).$$

5. (a) Let $u \in W_0^{1,2}$ satisfy

$$\int_\Omega \nabla u \cdot \nabla \phi\, dx \geq 0 \quad \forall \phi \in W_0^{1,2}, \quad \phi \geq 0.$$

   Show that $u \geq 0$ a.e. in Ω.

   (b) Let $u \in W_0^{1,2}$ satisfy the inequality in (a), show that

$$\inf_\Omega u \geq \inf_{\partial\Omega} u \quad \text{(essinf)}$$

109 學年度下學期博士班資格考試題
科目: 偏微分方程
2021 年 04 月 26 日

1. Solve the following initial boundary value problem

$$
\begin{aligned}
&4u_t = u_{xx}, && 0 < x < 1, \quad t > 0,\\
&u(x,0) = a_1 \sin(\pi x) + \cdots + a_m \sin(m\pi x), && 0 \le x \le 1,\\
&u(0,t) = u(1,t) = 0, && t > 0
\end{aligned}
$$

where $a_1, \cdots, a_m$ are constants and m is a positive integer.

2. State and prove the mean value property for harmonic functions.

3. (a) Let Ω be a bounded domain and Γ be a nonempty open subset of $\partial\Omega$ such that Γ is real analysis. Suppose that $\Delta u = 0$ in Ω, $u = 0, \nabla u = 0$ on Γ. Show that u is identically equal to zero in Ω.

   (b) State (without proof) carefully the theorems you used in (a)

4. Let K be a compact subset in $\mathbf{R}^n$. Define f to be uniform Hölder continuous with exponent $\alpha \in (0, 1]$ in K, (denoted by $f \in C^\alpha(K)$), if

$$\sup_{x,y\in K, x\neq y} \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \right\} < \infty.$$

Show that $fg \in C^\gamma(K)$ if $f \in C^\alpha(K)$ and $g \in C^\beta(K)$, where $\gamma = \min(\alpha, \beta)$.

5. Let

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

for all $x \in \mathbf{R}^n$ and $t > 0$.

   (a) Show that

$$\int_{\mathbf{R}^n} \Phi(x,t)\, dx = 1, \quad \forall t > 0.$$

   (b) Let $f \in C(\mathbf{R}^n) \bigcap L^\infty(\mathbf{R}^n)$ be a real-valued function, and

$$u(x,t) = \int_{\mathbf{R}^n} \Phi(x-y,t) f(y)\, dy.$$

   Show that, for any fixed $x \in \mathbf{R}^n$,

$$\lim_{t\to 0^+} u(x,t) = f(x).$$

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微分方程/PDEs

Total Time: 3 hours, November 03, 2023

1. **(a)** Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 2u = e^{x-2t}$$
with $u(x,0) = 0$. Find the explicit solution of this equation.

   (b) Solve the wave equation of $u : \mathbb{R}^2 \to \mathbb{R}$,
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^{2x},\ \forall\,(x,t) \in \mathbb{R}^2,$$
with the conditions, $u(x,0) = 0 = \partial_t u(x,0)$, $\forall\, t,x \in \mathbb{R}$.

2. Let
$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}},\ \forall\,(x,t) \in \mathbb{R} \times \mathbb{R}_+.$$
Assume that there exist constant numbers, $M > 0$ and $\alpha \in (0,1)$, such that the real-valued function $f \in C(\mathbb{R}\times\mathbb{R}) \cap L^\infty(\mathbb{R}\times\mathbb{R})$ fulfills
$$|f(x_2,t_2) - f(x_1,t_1)| \le M \cdot (|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2})$$
for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R}^2$. Let
$$z(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-y,t-\tau)\cdot f(y,\tau)\,dy\,d\tau, \qquad \forall\,(x,t) \in \mathbb{R}\times\mathbb{R}_+.$$

   (a) Show that $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x^2}$ are continuous in $\mathbb{R} \times (\mathbb{R}_+ \cup \{0\})$.

   (b) Give your answer with an argument on
$$\lim_{t\to 0} z(x,t) = ?$$

   (c) By considering the family of functions,
$$z_h(x,t) = \int_0^{t-h} \int_{-\infty}^{\infty} K(x-y,t-\tau)\cdot f(y,\tau)\,dy\,d\tau, \qquad \text{where } h \in (0,t/2),$$
show that $z(x,t)$ fulfills
$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f, \qquad \forall\,(x,t) \in \mathbb{R}\times\mathbb{R}_+.$$

3. Let $n \geq 3$ be an integer, and denote by $\mathbb{U}_R := \{x \in \mathbb{R}^n : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^n$, and by $\mathbb{B}_R := \{x \in \mathbb{R}^n : |x| \leq R\}$ the closed ball of radius $R > 0$ with center at the origin of $\mathbb{R}^n$. The Poisson representation formula is given by

$$u(x) = \frac{R^2 - |x|^2}{n\omega_n R} \int_{y \in \partial \mathbb{B}_R} \frac{g(y)}{|x-y|^n} \, dS(y), \ \forall \, x \in \mathbb{U}_R, \tag{1}$$

where $dS(y)$ represents the area element of $\partial \mathbb{B}_R$ at y, and ω_n is the volume of n-dimensional unit ball, and $x, y \in \mathbb{R}^n$. .

Answer the following questions with sufficient details.

(a) By applying the Poisson representation formula (1), prove the strong Maximum Principle of the harmonic function u, i.e., if

$$\sup_{\mathbb{B}_R} u = u(p), \text{ for some point } p \in \mathbb{U}_R,$$

then u is a constant function.

(b) Suppose $g \in C^{\alpha}(\mathbb{B}_R)$, for some $\alpha \in (0, 1)$. Show that the function u defined in (1) fulfills $u \in C^{2,\alpha}(\mathbb{B}_R)$, and

$$\Delta u = 0, \text{ in } \mathbb{U}_R,$$

$$\lim_{x \to x_0} u(x) = g(x_0), \ \forall \, x_0 \in \partial \mathbb{B}_R.$$

4. Assume $u : \mathbb{R}^n \to \mathbb{R}$ is a bounded non-zero harmonic function. Prove that

$$\int_{\mathbb{R}^n} u^2 \, dx$$

doesn't exist. (Hint: Use the formula of mean value property of harmonic functions on a ball, and apply Schwarz inequality to obtain a formula or an estimate on $\|u^2\|^2_{L^2(\mathbb{B}_R)}$. The formula of mean value property can be obtained from (1).)

5. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and simply-connected subset. Consider the weak solution in some Sobolev spaces to the elliptic partial differential equation of the form,

$$div(A(x, u, \nabla u)\nabla u) + B(x, u, \nabla u) = 0, \tag{2}$$

where $u : \Omega \to \mathbb{R}$ and $A(x, u, \nabla u)$ is a matrix valued function.

Pick at least one of the following questions to answer. A more complete argument is preferred.

(a) When the coefficients $A(x, u, \nabla u) = |\nabla u|^q \cdot id$ and $B(x, u, \nabla u) = f(x)$ belonging to the functional space L^∞, what is the proper Sobolev space to work with the existence problem? How kind of difficulties you may have in the regularity theory as applying linear theory of elliptic PDEs? What is the regularity result in the linear theory of elliptic PDEs you can use? Explain your answer.

(b) When the coefficients $A(x,u,\nabla u) = id$ and $B(x,u,\nabla u) = \Gamma(u)|\nabla u|^{\alpha}$ for some $\alpha \in (0,2]$ in Eq.(2), $W^{1,2}(\Omega)$ is a proper Sobolev space to work with. Can you briefly explain the strategy on how to apply the regularity theory of *linear* elliptic PDEs in Sobolev spaces. Note that you might discuss in two cases, $\alpha \in (0,2)$ and $\alpha = 2$.

(Hint: You might start from discussing the equation in terms of variational point of view, introduce the notion of weak solutions, and apply the regularity of a linear elliptic PDE with proper regularity of coefficients to the boot-strapping argument in nonlinear equations. The difficulties appear as you describe the regularity result of linear theory properly.)

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科目：偏微分方程
Math/NTNU Qualifying Exam of PDEs

April 30, 2024

1. (15 pt.) Let $u : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + u = e^{x-2t}$$

with $u(x,0) = 0$. Find the explicit solution of this equation.

2. (30 pt.) Denote by $\mathbb{B}_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball of radius $R > 0$ with center at the origin of $\mathbb{R}^d$. Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$u(x) = \frac{x}{|x|}.$$

(a) As $d = 2$, when is $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1,\infty)$? Give the range of p associated to your claim with explanation or argument.

(b) As $d = 1$,when is $u \in W^{1,p}(\mathbb{B}_1)$ for some $p \in [1,\infty)$? Give the range of p associated to your claim with explanation or argument.

Notice: Be sure to give an argument with sufficient details on why a function belongs to a Sobolev space.

3. (30 pt.)

Let

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4\,t}}, \forall\ (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

Assume that there exist constant numbers, $M > 0$ and $\alpha \in (0,1)$, such that the real-valued function $f \in C(\mathbb{R} \times \mathbb{R}) \cap L^\infty(\mathbb{R} \times \mathbb{R})$ fulfills

$$|f(x_2,t_2) - f(x_1,t_1)| \le M \cdot (|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2})$$

for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R}^2$. Let

$$z(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-y,t-\tau) \cdot f(y,\tau)\, dy\, d\tau.$$

Show with details that

(a) $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}, \frac{\partial^2 z}{\partial x^2}$ are continuous in $\mathbb{R} \times (\mathbb{R}_+ \cup \{0\})$.

(b) The equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f$$

holds in the domain $\mathbb{R} \times \mathbb{R}_+$.

Hint: Consider the family of functions,

$$z_h(x,t) = \int_0^{t-h} \int_{-\infty}^{\infty} K(x-y,t-\tau) \cdot f(y,\tau)\, dy\, d\tau,$$

where $h \in (0,t/2)$.

4. (25 pt.) Let $u \in C^0(\Omega)$ be a weakly harmonic function in an open and bounded set $\Omega \subset \mathbb{R}^n$, i.e.,

$$\int_\Omega u \Delta \varphi\, dx = 0$$

holds for all $\varphi \in C_0^2(\Omega)$. Show that u is harmonic in Ω.

Hints: Set $\rho(x) = Ce^{\frac{1}{|x|^2-1}}$ as $|x| < 1$, and $\rho(x) = 0$, otherwise. Here, C is some constant such that $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$. Consider the function, $u_\varepsilon := u * \rho_\varepsilon$, where $\rho_\varepsilon(y) = \varepsilon^{-n}\rho(y/\varepsilon)$. You are allowed to apply the mean value property of harmonic functions without giving the proof, but not quoting Weyl's Lemma.

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科目：偏微分方程
Math/NTNU Qualifying Exam of PDEs
November, 2024

1. (15 pt.) Find the explicit solution $u(x,t)$ of the following partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = -5\,u^2\,, x > 0, t > 0$$

with $u(x,0) = x^2, x > 0$.

2. Suppose that Ω is a bounded, connected open set in R^n. Prove the following properties respectively.

   (i) (20 pt.) If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω, then

   $$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u, \qquad \min_{\bar{\Omega}} u = \min_{\partial\Omega} u.$$

   (ii) (15 pt.) Let $f \in C(\Omega), g \in C(\partial\Omega)$ be two given functions. Then there is at most one solution of the following Dirichlet problem with $u \in C^2(\Omega) \cap C(\bar{\Omega})$:

   $$\begin{cases} -\Delta u = f \; in \; \Omega \\ \quad u = g \; on \; \partial\Omega. \end{cases}$$

3. (20 pt.) Assume $\varphi \in C(R^n) \cap L^\infty(R^n)$, and define u by

   $$u(x,t) = \frac{1}{(4k\pi t)^{n/2}} \int_{R^n} \varphi(y) e^{-\frac{|x-y|^2}{4kt}}\, dy \quad \text{for } t > 0.$$

   Prove the following properties:

   (i) $u \in C^\infty\big(R^n \times (0,\infty)\big)$.

   (ii) $u_t - k\Delta u = 0$ for all $x \in R^n, t > 0$.

   (iii) $\lim_{\substack{(x,t)\to(x_0,0) \\ x_0, x\in R^n, t>0}} u(x,t) = \varphi(x_0)$.

4. (20 pt.) Let Ω be an open, bounded set in R^n. Define the parabolic cylinder as $\Omega_T \equiv \Omega \times (0,T]$, the parabolic boundary of Γ_T as $\Gamma_T \equiv \overline{\Omega_T} - \Omega_T$. Assume u is a sufficiently smooth solution of the heat equation $u_t - k\Delta u = 0$ in Ω_T. Prove the following properties:

(i) $\max_{\Omega_T} u = \max_{\Gamma_T} u$. (Weak maximum principle)

(ii) If Ω is connected and there exists a point $(x_0, t_0) \in \Omega_T$ such that

$$u(x_0, t_0) = \max_{\Omega_T} u(x,t),$$

then u is constant in $\overline{\Omega_{t_0}}$. (Strong maximum principle)

5. (20 pt.) Assume $f \in C^2(R), g \in C^1(R)$ and define u by

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy \text{ for } x \in R, t \geq 0.$$

Prove the following properties:

(i) $u \in C^2\left(R \times [0,\infty)\right), u_{tt} - u_{xx} = 0$ in $R \times (0,\infty)$.

(ii) $\lim_{\substack{(x,t)\to(x^0,0) \\ t>0}} u(x,t) = f(x^0)$, $\lim_{\substack{(x,t)\to(x^0,0) \\ t>0}} u_t(x,t) = g(x^0)$

for each point $x^0 \in R$.