100 學年度下學期數學系博士班資格考試
(實變分析)

本試題卷共 2 頁，計 10 題計算證明題，每題 10 分，合計 100 分。

1. Prove the *Carathéodory theorem*: A set E is measurable if and only if for every set A,

$$|A|_e = |A \cap E|_e + |A \setminus E|_e.$$

   (Note: $|A|_e|$ denotes the outer measure of A.)

2. Prove that the set of points at which a sequence of measuable real-valued functions converges (to a finite limit) is measurable.

3. Let f be a function which is upper semi-continuous and finite on a compact set E. Show that if f is bounded above on E. Show also that f assumes its maximum on E, that is, that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

4. Let $f \in L(0,1)$. Show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and $\int_0^1 x^k f(x)\,\mathrm{d}x \to 0$ as $k \to \infty$.

5. Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y \mid (x,y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero, and the for almost every $y \in \mathbb{R}^1$, $\{x \mid (x,y) \in E\}$ has measure zero.

6. (a) Write out the definition of the essential supremum $\|f\|_\infty$ of a real-valued measurable function f on a measurable set E.

   (b) Let f be a real-valued measurable function on $[0,1]$. Prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

7. Let E be a measurable set in $\mathbb{R}^n$, and $0 < p < q \leq \infty$.

   (a) Prove that $L^p(E) \cap L^\infty(E) \subset L^q(E)$.

   (b) Prove that if $|E| < \infty$, then $L^q(E) \subset L^p(E)$.

8. Let $f, g \in L^2(\mathbb{R}^n)$. Prove that $f + g \in L^2(\mathbb{R}^n)$ and $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

(背面尚有試題)

9. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0,1]$, and $\{c_k\}$ be the Fourier series of a function $f \in L^2[0,1]$ with respect to the system $\{\varphi_k\}$.

   (a) Prove that the Bessel's inequality $\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2} \leq \|f\|_2$ holds.

   (b) Find a necessary and sufficient condition so that the Parseval's identity $\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2} = \|f\|_2$ holds, and prove your answer.

10. Let $C[0,1]$ denote the set of all real-valued continuous functions on $[0,1]$, and the linear operator $T : C[0,1] \to \mathbb{R}$ be defined by $T(f) = f(1)$ for all $f \in C[0,1]$.

   (a) Prove that T is a continuous linear functional on $C[0,1]$.

   (b) Prove that there exists an extension $T^* : L^\infty[0,1] \to \mathbb{R}^n$ of T such that T^* is a continuous linear functional on $L^\infty[0,1]$, but there is no $g \in L^1[0,1]$ satisfying

$$T^*(f) = \int_{[0,1]} (f \times g)\, \mathrm{d}x \qquad \text{for all } f \in C[0,1].$$

(試題結束)

101 學年度上學期數學系博士班資格考試
(實變分析)

本試題卷共 2 頁，計 10 題計算證明題，每題 10 分，合計 100 分。

1. Let E be a measurable subset of $\mathbb{R}$, with $|E| > 0$. Prove that there exists a positive real number ε such that $(-\varepsilon, \varepsilon) \subset E - E$, where
$$E - E = \{x - y \mid x, y \in E\}.$$

2. Prove or disprove:
   (a) Any function $f : [a,b] \to \mathbb{R}$ of bounded variation is measurable.
   (b) Any upper semicontinuous function $f : [a,b] \to \mathbb{R}$ is measurable.

3. Let E be a measurable set in $\mathbb{R}^n$ of finite measure. Prove that $f : E \to \mathbb{R}$ is measurable if and only if for any $\varepsilon > 0$, there exists a closed subset F of E such that $|E \setminus F| < \varepsilon$, and f is continuous on F.

4. (a) State without proof the Egorov's theorem.
   (b) Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable set E with $|E| < \infty$. If f_k converges to f a.e. in E, and $\sup_k |f_k - f| \in L(E)$, prove that $\lim_{k\to\infty} \int_E f_k = \int_E f$.

5. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ satisfy for each $x \in [0,1]$, $f(x,y)$ is a Lebesgue integrable function of y, and $\dfrac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Prove that $\dfrac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each $x \in [0,1]$, and
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{[0,1]} f(x,y)\,\mathrm{d}y = \int_{[0,1]} \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y.$$

6. (a) State the definition for a finite function f on a finite interval $[a,b]$ to be *absolutely continuous*.
   (b) Show that the function $f(x) = x^{\alpha}$ is absolutely continuous on every bounded subinterval of $[0,\infty)$ whenever $\alpha > 0$.

7. Let $a_1, a_2, \ldots, a_N$ be non-negative real numbers, $p_1, p_2, \ldots, p_N$ be positive real numbers with $\sum_{j=1}^{N}(1/p_j) = 1$. Show that
$$\prod_{j=1}^{N} a_j \le \sum_{j=1}^{N} \frac{a_j}{p_j}.$$

(背面尚有試題)

8. Let ℓ^∞ denote the normed linear space of all bounded real sequences. Is ℓ^∞ separable? Justify your answer.

9. Suppose that $f_k, f \in L^2$, and that $\int f_k g \to \int f g$ for all $g \in L^2$. If $\|f_k\|_2 \to \|f\|_2$, show that $f_k \to f$ in L^2 norm.

10. Let Σ be a σ-algebra on a set $\mathscr{S}$, $\{E_k\}$ be any sequence of sets in Σ, and ϕ be a non-negative additive set function on Σ. Prove that

$$\phi\left(\liminf_{k\to\infty} E_k\right) \le \liminf_{k\to\infty} \phi(E_k).$$

(試題結束)

103 學年度數學系博士班資格考試

（實 變 分 析）

※ 本試題卷共 8 題證明題

1. (a) Prove that every Borel measurable subset in $\mathbb{R}^n$ is Lebesgue measurable.

(b) Prove that there is a Lebesgue measurable subset in $\mathbb{R}^n$ is not Borel measurable.

(10%)

2. Prove or disprove (Please explain your answer):

(a) If $f:[a,b]\to\mathbb{R}$ is a function of bounded variation, then f is Lebesgue measurable.

(b) If E is a Lebesgue measurable subset of $\mathbb{R}$, with $|E|>0$, then there exist $x,\ y\in E$ with $x\neq y$ such that $x-y$ is a rational number.

(c) If for each rational number a, the set $\{x\in\mathbb{R}^n \mid f(x)>a\}$ is Lebesgue measurable, then $f:\mathbb{R}^n\to\mathbb{R}$ is Lebesgue measurable.

(d) There exists a Riemann integrable function $f:[0,1]\to[0,1]$ such that f is continuous at each rational point and discontinuous at each irrational point of $[0,1]$.

(e) If f is Lebesgue integrable over E, then f is finite a.e. in E. (30%)

3. Prove that if $f:[a,b]\to\mathbb{R}$ is a function of bounded variation, then f can be written as $f = g + h$, where g is absolutely continuous and h is singular, which are unique up to additive constants. (10%)

4. Prove that if $f \in L^p(E)$ and $f\geq 0$, then $\int_E f^p = p\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$, where ω is the distribution function of f, defined by $\omega(\alpha) = |\{x\in E \mid f(x)>\alpha\}|$. (10%)

5. Prove that if $f \in L^p(\mathbb{R})$, where $1\leq p<\infty$, then for every $\varepsilon>0$ there is a continuous function g with compact support such that $\|f-g\|_p < \varepsilon$. (10%)

6. Prove that if $f\in L(\mathbb{R}^n)$, then the definite integral $F(E) = \int_E f(x)\,dx$ is absolutely continuous with respect to Lebesgue measure. (10%)

7. For $f, g \in L(\mathbb{R}^n)$, we define the convolution of f and g by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \quad \text{for } x \in \mathbb{R}^n .$$

Prove that $f * g \in L(R^n)$, and $\|f * g\|_1 \le \|f\|_1 \cdot \|g\|_1$. (10%)

8. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0,1]$, and $\{c_k\}$ be a sequence in $\ell^2(R)$. Prove that there exists $f \in L^2[0,1]$ such that $\sum_{k=1}^{\infty} c_k \varphi_k(x)$ is the Fourier series of f with respect to the orthonormal system $\{\varphi_k\}$. (10%)

103 學年度數學系博士班資格考試

(實 變 分 析)

2015.4.30

※ 本試題卷共 8 題計算證明題

1. (a) Prove that if every measurable set E in $\mathbb{R}^n$ can be expressed as $E = F \cup Z$, where F is a closed set and $|Z| = 0$.

   (b) Let E_1 and E_2 be measurable subsets of $\mathbb{R}^n$. Prove that the product set $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^n \times \mathbb{R}^n$, and $|E_1 \times E_2| = |E_1| \cdot |E_2|$.

(15%)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Prove that the function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $g(x, y) = f(x - y)$ is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. (10%)

   Hint : Show that there exists an invertible (2×2) matrix A such that

   $$\{ (x, y) \mid g(x, y) > a \} = A\left(\mathbb{R}^n \times \{ z \mid f(z) > a \}\right) \text{ for all } a \in \mathbb{R}.$$

3. Prove or disprove (Please explain your answer):

   (a) There exists a Riemann integrable function $f : [0,1] \to [0,1]$ such that f is continuous at each rational point and discontinuous at each irrational point of $[0,1]$.

   (b) There exists an increasing continuous function f whose derivative f' is Lebesgue integrable on $[0,1]$ such that $\int_{[0,1]} f' \neq f(1) - f(0)$. (10%)

4. (a) Prove carefully that for $0 < a < b < \infty$, $\int_{[0,\infty)} \int_{[a,b]} e^{-xy} \sin x \, dx \, dy = \int_{[a,b]} \frac{\sin x}{x} \, dx$.

   (b) Evaluate the Lebesgue integral $\int_{(0,\infty)} \frac{\sin x}{x} \, dx$. (15%)

5. Let $f : [0,1] \to \mathbb{R}$ be measurable. Prove that if $g(x, y) = f(x) - f(y)$ is Lebesgue integrable over $[0,1] \times [0,1]$, then f is Lebesgue integrable on $[0,1]$.

(10%)

6. Let $f_k : E \to \mathbb{R}$ be a sequence of measurable functions on E, where E is a measurable subset of $\mathbb{R}^n$, and $1 \le p < \infty$.

(a) State the definition that $\langle f_k \rangle$ converges to f in measure.

(b) State the definition that $\langle f_k \rangle$ converges to f in L^p.

(c) Prove that if $\langle f_k \rangle$ converges to f in L^p, then it converges to f in measure.

(15%)

7. (a) State without proof Holder inequality.

(b) Let E be a measurable subset of $\mathbb{R}^n$, with $|E| \le 1$, and $1 \le p < q < \infty$. Prove that for any measurable function $f : E \to \mathbb{R}$, $\|f\|_p \le \|f\|_q$.

(10%)

8. (a) Let $f \in L^2(0,1)$. Prove that $\lim_{k\to\infty} \int_0^{2\pi} f(x)\cos kx\,dx = \lim_{k\to\infty} \int_0^{2\pi} f(x)\sin kx\,dx = 0$.

(b) Is (a) still true if $f \in L^1(0,1)$? Why?

(15%)

104 學年度數學系博士班資格考試

(實 變 分 析)

2015. 10. 30

※ 本試題卷共六大題(第一大題 50 分，其餘各題每題 10 分)

1. Prove or disprove： (Please explain your answer)

(1) There is a Lebesgue measurable subset in $\mathbb{R}^n$, which is not Borel measurable.

(2) Any function f of bounded variation on $[a,b]$ is Riemann integrable .

(3) There is a subset E of $\mathbb{R}$, with $|E|_e > 0$, satisfying for any $x,\ y \in E$ with $x \neq y$, $x - y$ is not a rational number.

(4) There is a sequence $\{E_k\}$ of disjoint sets such that $\left|\bigcup_{k=1}^{\infty} E_k\right|_e < \sum_{k=1}^{\infty} |E_k|_e$.

(5) If $f:\mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable, then the function $g:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $g(x,\ y) = f(x - y)$ is also Lebesgue measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

(6) Every Riemann integrable function $f:[0,1] \to \mathbb{R}$ is Lebesgue integrable.

(7) If f is Lebesgue integrable over E, then f is finite a.e. in E.

(8) If $1 \le p < q < \infty$, then $L^q[0,1] \subset L^p[0,1]$.

(9) There exists an increasing continuous function f whose derivative f' is Lebesgue integrable on $[0,1]$ such that $\int_{[0,1]} f' \neq f(1) - f(0)$.

(10) Any function f of bounded variation on $[a,b]$ can be written as $f = g + h$, where g is absolutely continuous and h is singular.

(50%)

2. Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be an affine function defined by $T(x) = Ax + u$, where A is an $n \times n$ matrix, and u is a fixed vector in $\mathbb{R}^n$. Prove that for any Lebesgue measurable set E of $\mathbb{R}^n$, $|T(E)| = |\det A||E|$. (10%)

3. Let $f:E \to \mathbb{R}$ be a Lebesgue measurable function, where E is a Lebesgue measurable subset of $\mathbb{R}^n$ with $|E| < \infty$. Prove that there exists a sequence $\langle f_k \rangle$ of simple measurable functions on E such that $\langle f_k \rangle$ converges almost uniformly to f in the following sense: for all $\varepsilon > 0$, there exists a closed subset F of E with $|E \setminus F| < \varepsilon$, such that $\langle f_k \rangle$ converges uniformly to f on F. (Hint: You can apply Egorov Theorem) (10%)

4. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ satisfy for each $x \in [0, 1]$, $f(x, y)$ is a Lebesgue integrable function of y, and $\frac{\partial f(x, y)}{\partial x}$ is a bounded function of (x, y). Prove that $\frac{\partial f(x, y)}{\partial x}$ is a Lebesgue measurable function of y for each $x \in [0, 1]$, and

$$\frac{d}{dx} \int_{[0,1]} f(x, y)\, dy = \int_{[0,1]} \frac{\partial f(x, y)}{\partial x}\, dy .$$

(10%)

5. Let f be nonnegative and Lebesgue measurable on a Lebesgue measurable subset E of $\mathbb{R}^n$. Prove that

$$\int_E f = \sup \sum_j [\inf_{x \in E_j} f(x)] |E_j| ,$$

where the supremum is taken over all decompositions $E = \cup_j E_j$ of E into the union of a finite number of disjoint Lebesgue measurable sets E_j. (10%)

6. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$, and $\{c_k\}$ be a sequence in $\ell^2(\mathbb{R})$. Prove that there exists $f \in L^2[0, 1]$ such that $\sum_{k=1}^{\infty} c_k \varphi_k(x)$ is the Fourier series of f with respect to the orthonormal system $\{\varphi_k\}$. (10%)

105 學年度數學系博士班資格考試

(Real Analysis Qualifying Exam) **2016.10.31**

1. Let E, F be measurable sets in $\mathbb{R}^n$, B be a Borel set in $[0,\infty)$, and $f: E \to [0,\infty)$ be a measurable function. Prove that the following 4 sets are measurable:

$$E \cup F,\ E \times F,\ f^{-1}\{B\},\ \text{and}\ R(f,E) = \{(x,y) \mid x \in E, 0 \le y \le f(x)\}. \qquad (20\%)$$

2. (a) Use Caratheodory theorem to show that if E is a subset of $\mathbb{R}^n$ satisfying the condition $|G| = |G \cap E|_e + |G \cap E^C|_e$ for all open sets G in $\mathbb{R}^n$, then E is measurable.

   (b) If the condition in (a) is changed to $|F| = |F \cap E|_e + |F \cap E^C|_e$ for all closed sets F in $\mathbb{R}^n$, is E measurable? Why? (10%)

3. Prove that if $f: \mathbb{R}^n \to \mathbb{R}$ is a measurable function, then the function $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by $g(x, y) = f(2x - 3y)$, is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. (10%)

   (Hint: Find an invertible (2×2) matrix A such that

$$\{ (x, y) \mid g(x, y) > a \} = A\left(\mathbb{R}^n \times \{ z \mid f(z) > a \}\right) \text{ for every } a \in \mathbb{R}.)$$

4. Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable set E of $\mathbb{R}^n$.

   (a) Use monotone convergence theorem to show that $\int_E \sum_{k=1}^{\infty} |f_k| = \sum_{k=1}^{\infty} \int_E |f_k|$.

   (b) Prove that if the series $\sum_{k=1}^{\infty} \int_E |f_k|$ converges, then $\sum_{k=1}^{\infty} f_k$ converges absolutely *a.e.* in E, and $\sum_{k=1}^{\infty} \int_E f_k = \int_E \sum_{k=1}^{\infty} f_k$. (16%)

5. (a) Prove that if $f \in L(E)$, then for all $\varepsilon > 0$, there is $\delta > 0$ such that $\int_A |f| < \varepsilon$ for all measurable subsets A of E with $|A| < \delta$.

   (b) Use Egoroff theorem to show that if $\langle f_k \rangle$ is a sequence of measurable functions that converges to f a.e. in E, with $|E| < \infty$, and $\sup_k |f_k - f| \in L(E)$, then $\lim_{k\to\infty} \int_E f_k = \int_E f$.

   (c) Use Tonelli theorem to show that if $f, g \in L(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} |f(x-y) \times g(y)|\, dy < \infty$ for a.e. $x \in \mathbb{R}^n$. (24%)

6. Let $\{\varphi_k\}$ be an orthonormal system in $L^2[0, 1]$. Prove that $\{\varphi_k\}$ is complete if, and only if, Parseval's formula $\|f\| = \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2}$ holds for every $f \in L^2[0, 1]$, where the numbers c_k are the Fourier coefficients of f with respect to the system $\{\varphi_k\}$. (10%)

7. Use Radon-Nikodym theorem to show that for any continuous linear functional T on $L^2[0, 1]$, there exists a unique function $g \in L^2[0, 1]$ such that $T(f) = \int_{[0,1]} f \times g$ for every $f \in L^2[0, 1]$. (10%)

106 學年度數學系博士班資格考試

(Real Analysis Qualifying Exam) **2017.10.31**

*****Each problem is worth 10 points.*****

1. Determine which function is Riemann (improper) integrable on E ? Lebeague integrable on E ? Explain your answer.
$$f(x)=\begin{cases}1, & \text{if } x\in[0,1]\cap\mathbb{Q}\\ x, & \text{if } x\in[0,1]\cap\mathbb{Q}^C\end{cases} \text{ on } E=[0,1] \text{ and } g(x)=\frac{\sin x}{x} \text{ on } E=[1,\infty).$$

2. Prove that (Caratheodory Theorem) a subset E in $\mathbb{R}^n$ is measurable if and only if for every set A in $\mathbb{R}^n$, $|A|_e=|A\cap E|_e+|A\setminus E|_e$.

3. Construct a sequence of disjoint sets $E_1,E_2,E_3,\cdots$ in $\mathbb{R}$ such that $\left|\bigcup_{k=1}^{\infty}E_k\right|_e\neq\sum_{k=1}^{\infty}|E_k|_e$.

4. Prove that there exists a Lebesgue measurable set in $\mathbb{R}$, which is not a Borel set.

5. Prove that if $f:\mathbb{R}^n\to\mathbb{R}$ is measurable, then the function $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, defined by $g(x,y)=f(x+2y)$, is also measurable on $\mathbb{R}^n\times\mathbb{R}^n$.

6. Let $\langle f_k\rangle$ be a sequence of measurable functions on a measurable set E of $\mathbb{R}^n$. Prove that if the series $\sum_{k=1}^{\infty}\int_E|f_k|$ converges, then $\sum_{k=1}^{\infty}f_k$ converges absolutely *a.e.* in E, and $\sum_{k=1}^{\infty}\int_E f_k=\int_E\sum_{k=1}^{\infty}f_k$.

7. Suppose that $f\in L(\mathbb{R})$ and $\iint_{\mathbb{R}^2}f(3x)f(x+2y)\,dx\,dy=1$, calculate $\int_{\mathbb{R}}f(x)\,dx$.

8. (a) Prove that if $f:[a,b]\to\mathbb{R}$ is bounded, Lebesgue integrable, and $F(x)=\int_{[a,x]}f$, then F is absolutely continuous, and $F'=f$ *a.e.* in $[a,b]$.

   (b) Is (a) still true, if f is unbounded? Why?

9. Let $f\in L^p(\mathbb{R}^n)$, $1<p,q<\infty$, and $\frac{1}{p}+\frac{1}{q}=1$. Prove that $\|f\|_p=\sup_{\|g\|_q\le 1}\left|\int_{\mathbb{R}^n}f(x)\times g(x)\,dx\right|$.

10. (a) Let $f\in L^2(0,2\pi)$. Prove that $\lim_{k\to\infty}\int_0^{2\pi}f(x)\cos kx\,dx=\lim_{k\to\infty}\int_0^{2\pi}f(x)\sin kx\,dx=0$.

    (b) Is (a) still true, if $f\in L^1(0,2\pi)$? Why?

108 學年度數學系博士班資格考試(實變分析)

Real Analysis Qualifying Exam 2019.10.31

1. It is known from Caratheodory theorem that a subset E of $\mathbb{R}^n$ is measurable if and only if $|A| = |A \cap E|_e + |A \setminus E|_e$ for all sets A in $\mathbb{R}^n$. Prove or disprove :
   (a) If $|G| = |G \cap E|_e + |G \setminus E|_e$ for all open sets G in $\mathbb{R}^n$, then E is measurable.
   (b) If $|F| = |F \cap E|_e + |F \setminus E|_e$ for all closed sets F in $\mathbb{R}^n$, then E is measurable.

(12%)

2. (a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and B denote the Borel σ-algebra in $\mathbb{R}$. Prove that the family $\Gamma = \{E \subset \mathbb{R} \mid f^{-1}(E) \text{ is measurable}\}$ is a σ-algebra containing B.
   (b) Prove that there exists a measurable subset of $[0,1]$, but not a Borel set.

(12%)

3. (a) Prove that every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps measurable subsets of $\mathbb{R}^n$ into measurable sets.
   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function , and $a, b \in \mathbb{R}$. Prove that the function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by $g(x, y) = f(ax + by)$, is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

(12%)

4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then f must be linear. (10%)

5. (a) Prove that if $f \in L(E)$, then f is finite *a.e.* in E.
   (b) Suppose that $\langle f_k \rangle$ is a sequence of measurable functions on a measurable subset E of $\mathbb{R}^n$, and $\sum_{k=1}^{\infty} \int_E |f_k|$ converges. Prove that $\sum_{k=1}^{\infty} f_k$ converges absolutely *a.e.* in E, and $\sum_{k=1}^{\infty} \int_E f_k = \int_E \sum_{k=1}^{\infty} f_k$. (12%)

6. Let $\langle f_k \rangle$ be a sequence of measurable functions on a measurable subset E of $\mathbb{R}^n$, with $|E| < \infty$, and $|f_k(x)| \le M_x < \infty$ for all k and for each $x \in E$. Prove that for all $\varepsilon > 0$, there is a closed subset F of E and a positive number M such that $|E \setminus F| < \varepsilon$ and $|f_k(x)| \le M$ for all k and for all $x \in F$. **(Hint :** You can apply Lusin theorem) (10%)

7. Use Tonelli theorem to show that if $f : E \to [0,\infty)$ is a measurable function on a measurable subset E of $\mathbb{R}^n$, and $\omega(\alpha) = \left|\left\{ x \in E \mid f(x) > \alpha \right\}\right|$, then $\int_E f = \int_0^\infty \omega(\alpha)\, d\alpha$.

(**Hint :** $\int_E f = \iint_{R(f,E)} 1\, dx\, dy$, where $R(f,E) = \{(x,y) \mid x \in E, 0 \le f(x) \le y\}$.) (10%)

8. Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be a measurable function. Prove that if the iterated integral $\int_{[0,1]}\int_{[0,1]} |f(x,y)|\, dx\, dy$ exists and is finite, then $f \in L([0,1]\times[0,1])$, and

$$\iint_{[0,1]\times[0,1]} f = \int_{[0,1]}\int_{[0,1]} f(x,y)\, dx\, dy = \int_{[0,1]}\int_{[0,1]} f(x,y)\, dy\, dx .$$

(10%)

9. Let $\{\varphi_k\}$ be any orthonormal basis for $L^2(E)$ over $\mathbb{R}$.

(a) Prove that $\{\varphi_k\}$ must be countable and complete.

(b) Prove that any function $f \in L^2(E)$ satisfies Parseval formula with respect to $\{\varphi_k\}$; that is, $\|f\|_2 = \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{\frac{1}{2}}$, where $\{c_k\}$ is the sequence of Fourier coefficients of f.

(12%)

109 學年度數學系博士班資格考試(實變分析)

Real Analysis Qualifying Exam **2021.4.28**

1. Let $f(x)=\begin{cases}0, & \text{if } x\in[0,1]\\ 1, & \text{if } x\in(1,2]\end{cases}$, $\alpha(x)=\begin{cases}0, & \text{if } x\in[0,1)\\ 1, & \text{if } x\in[1,2]\end{cases}$, and $\beta(x)=\begin{cases}x, & \text{if } x\in[0,1)\\ x^2, & \text{if } x\in[1,2]\end{cases}$.

   (a) Is f Riemann-Stieltjes integrable to α on $[0,2]$? Why?

   (b) Is f Riemann-Stieltjes integrable to β on $[0,2]$? Why? (12%)

2. (a) Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be a measurable function and B be a Borel set in $\mathbb{R}$. Prove that $f^{-1}(B)$ is measurable in $[0,1]\times[0,1]$.

   (b) Let f and g be measurable on $[0,1]$. Prove that the function $F:[0,1]\times[0,1]\to\mathbb{R}$, defined by $F(x,y)=f(x)\times g(y)$, is measurable on $[0,1]\times[0,1]$. (12%)

3. Let $f:E\to\mathbb{R}$ be a measurable function on a measurable subset E of $\mathbb{R}^n$. Prove that for all $\varepsilon>0$, there is a Borel set B in E, with $|E\setminus B|<\varepsilon$, and a sequence $\langle f_k\rangle$ of Borel measurable functions such that $\langle f_k(x)\rangle$ converges increasingly to $|f(x)|$ for all $x\in B$.

   (10%)

4. Let $\langle f_k\rangle$ be a sequence of measurable functions on a measurable subset E of $\mathbb{R}^n$, and $\sum_{k=1}^{\infty}\int_E|f_k|$ converges. Prove that $\sum_{k=1}^{\infty}|f_k|$ converges *a.e.* in E, and $\sum_{k=1}^{\infty}\int_E f_k = \int_E\sum_{k=1}^{\infty} f_k$.

   (10%)

5. Let $\langle f_k\rangle$ be a sequence of increasing functions on $[a,b]$, and $\sum_{k=1}^{\infty} f_k(x)$ converge to $f(x)$ for each $x\in[a,b]$. Prove that $\sum_{k=1}^{\infty} f_k'(x)$ converges to $f'(x)$ for *a.e.* x in E.

   (10%)

6. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ satisfy that for each $x \in [0,1]$, $f(x, y)$ is a Lebesgue integrable function of y, and $\frac{\partial f(x,y)}{\partial x}$ is a bounded function of (x, y). Prove that $\frac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each $x \in [0,1]$, and

$$\frac{d}{dx}\int_{[0,1]} f(x,y)\,dy = \int_{[0,1]} \frac{\partial f(x,y)}{\partial x}\,dy. \qquad (10\%)$$

7. Let E be a measurable subset of $\mathbb{R}^n$. Prove that $f : E \to \mathbb{R}$ is measurable if and only If the region $R(f, E)$ is measurable, where $R(f,E) = \{(x,y) \mid x \in E, 0 \le f(x) \le y\}$.

(12%)

8. (a) Let f be measurable on E, and $1 < p < q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$\int_E |fg| \le \left(\int_E |f|^p\right)^{\frac{1}{p}} \left(\int_E |f|^q\right)^{\frac{1}{q}}$$

(b) Let f be measurable on E with $0 < |E| < \infty$, and $1 \le p < q < \infty$. Prove that

$$\left(\frac{1}{|E|}\int_E |f|^p\right)^{\frac{1}{p}} \le \left(\frac{1}{|E|}\int_E |f|^q\right)^{\frac{1}{q}}. \qquad (12\%)$$

9. Define the operator $T : C[0,1] \to \mathbb{R}$ by $T(f) = f(1)$ for all $f \in C[0,1]$, where $C[0,1]$ denotes the Banach space of all real-valued continuous functions on $[0,1]$.

(a) Prove that T is a continuous linear functional on $C[0,1]$.

(b) Prove that there exists a continuous linear functional $T^* : L^\infty[0,1] \to \mathbb{R}$ such that $T^*(f) = T(f)$ for all $f \in C[0,1]$, but there exists no function $g \in L^1[0,1]$ satisfying $T^*(f) = \int_{[0,1]} (f \times g)\,dx$ for all $f \in C[0,1]$. (12%)

Real Analysis Qualifying Exam

Fall 112.

English Name: ______________________________

Chinese Name: ______________________________

Grading. The exam is out of 100pts. As written below, Problems 1, 2, 6, 7, 8 are worth 12 pts; Problems 4, 5 are worth 13 pts; Problem 3 is worth 14 pts.

Preliminaries. Throughout this exam, we suppose X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X, elements of which we call measurable, and $\mu : \mathcal{B} \to [0, \infty]$ is a measure:

i
$$\mu(\emptyset) = 0;$$

ii
$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n) \quad E_n \in \mathcal{B} \text{ for all } n, E_n \cap E_m = \emptyset \text{ if } m \neq n.$$

Further suppose $X = \cup_n X_n$ with $\mu(X_n) < +\infty$. We say that a function $f : X \to [-\infty, \infty]$ is measurable if $\{x : f(x) > \alpha\} \in \mathcal{B}$ for each $\alpha \in \mathbb{R}$. For a measurable function $f : X \to [0, \infty]$ define

$$\int_X f \, d\mu := \sup_{g \leq f} \int_X g \, d\mu$$

where the supremum is taken over all non-negative simple functions.

1 (12 pts). Suppose that $f : X \to \mathbb{R}$ is a measurable function such that

$$\int_X |f| \, d\mu < +\infty.$$

Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if A is a measurable set with $\mu(A) < \delta$ then

$$\int_A |f| \, d\mu < \epsilon. \tag{1}$$

2 (12 pts). Show that if $\{A_n\}$ is a sequence of measurable sets with $A_{n+1} \subset A_n$ and $\mu(A_1) < +\infty$, then

$$\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=0}^{\infty} A_n). \tag{2}$$

3 (14 pts). Show that if $f_n : X \to [0, \infty]$ is a sequence of measurable functions such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$, then

$$\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu. \tag{3}$$

(If you utilize Egorov's theorem, monotone convergence theorem, dominated convergence theorem, etc. in your proof you should prove them first.)

4 (13 pts). In this problem, let $X = \mathbb{R}^n$ and suppose μ is a Radon measure, i.e. finite on compact sets and for each measurable set satisfies

$$\mu(E) = \sup_{K \subset E} \mu(K) = \inf_{U \supset E} \mu(U)$$

where K are assumed to be compact and U open. Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is integrable then there exists a sequence of continuous functions φ_n such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |\varphi_n - f| \, dx = 0. \tag{4}$$

5 (13 pts). In this problem, let $X = \mathbb{R}^n$ and suppose μ is a Radon measure, i.e. finite on compact sets and for each measurable set satisfies

$$\mu(E) = \sup_{K \subset E} \mu(K) = \inf_{U \supset E} \mu(U)$$

where K are assumed to be compact and U open. Define the Hardy-Littlewood maximal function of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ which is integrable on compact subsets by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, d\mu.$$

Suppose for the given μ that one has shown the weak-type estimate

$$\mu(\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > t\}) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f| \, d\mu.$$

Use this estimate and the properties of μ to show that

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) \, d\mu = f(x) \tag{5}$$

for μ almost every $x \in \mathbb{R}^n$. (You may assume that the conclusion of Problem 4 is valid.)

6 (12 pts). In this problem, let $X = [0,1]$, $\mathcal{B} = \mathcal{M}$ be the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and μ be the Lebesgue measure. Suppose that $f_n, f \in L^2([0,1])$,

$$\lim_{n\to\infty} \int_{[0,1]} f_n g \, dx = \int_{[0,1]} f g \, dx$$

for every $g \in L^2([0,1])$ and that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n|^2 \, dx = \int_{[0,1]} |f|^2 \, dx.$$

Show that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n - f|^2 \, dx = 0. \tag{6}$$

7 (12 pts). In this problem, let $X = [0,1]$, $\mathcal{B} = \mathcal{M}$ be the σ-algebra of Lebesgue measurable subsets of $[0,1]$ and μ be the Lebesgue measure. Suppose that $f_n, f \in L^2([0,1])$ and

$$\lim_{n\to\infty} \int_{[0,1]} |f_n - f|^2 \, dx = 0.$$

Show that there exists a subsequence $\{f_{n_k}\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \tag{7}$$

for Lebesgue almost every $x \in [0,1]$.

8 (12 pts). Let ν be another measure on the measurable space $(X, \mathcal{B})$ for which $X = \cup_n X'_n$ with $\nu(X'_n) < +\infty$. State the Radon-Nikodym theorem and the Lebesgue decomposition theorem for the measures μ, ν, introducing suitable hypothesis when necessary.

Real Analysis Qualifying Exam

Spring 113.

English Name: ____________________________

Chinese Name: ____________________________

Grading. The exam is out of 100pts. All problems are worth 20pts.

1 (20 pts) Suppose $1 \leq p < +\infty$ and let $L^p([0,1])$ denote the vector space of Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_{L^p([0,1])} := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

is finite.

1. Show that $f \mapsto \|f\|_{L^p([0,1])}$ is a norm.
2. Show that $L^p([0,1])$ is complete.
3. Show that continuous functions are dense in $L^p([0,1])$.

2 (20 pts) Define the Hardy-Littlewood maximal function (with respect to the Lebesgue measure) of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ which is integrable on compact subsets by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dx$$

Prove the weak-type estimate

$$|\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > t\}| \leq \frac{C}{t} \int_{\mathbb{R}^n} |f| \, dx.$$

3 (20 pts). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $|f|^p$ has finite integral. Prove that

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = p \int_0^\infty t^{p-1} |\{x \in \mathbb{R}^n : |f(x)| > t\}| \, dt.$$

4 (20 pts). Show that if $f_k : \mathbb{R}^n \to [0,\infty]$ is a sequence of measurable functions such that

$$f(x) = \lim_{n\to\infty} f_k(x)$$

exists for every $x \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f \, dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} f_k \, dx. \tag{1}$$

(If you utilize Egorov's theorem, monotone convergence theorem, dominated convergence theorem, etc. in your proof you should prove them first.)

5 (20 pts) For $1 \leq p < +\infty$, let l^p denote the space of sequences $a = \{a_n\}_{n\in\mathbb{N}}$ such that

$$\|a\|_{l^p} := \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$$

is finite. For a fixed $1 \leq p < +\infty$, let L be a linear functional on l^p, i.e., suppose L satisfies

$$L(\alpha a + \beta b) = \alpha L(a) + \beta L(b)$$

for all $\alpha, \beta \in \mathbb{R}$, $a = \{a_n\}_{n\in\mathbb{N}}, b = \{b_n\}_{n\in\mathbb{N}} \in l^p$ and there exists a constant $C = C(L) > 0$ such that

$$|L(a)| \leq C\|a\|_{l^p}.$$

1. Show that if $1 < p < +\infty$ we may identify $L = b$ for some $b = \{b_n\}_{n\in\mathbb{N}} \in l^{p/(p-1)}$, i.e. show there exists $b = \{b_n\}_{n\in\mathbb{N}} \in l^{p/(p-1)}$ such that

$$L(a) = \sum_{n=1}^{\infty} a_n b_n \tag{2}$$

for every $a = \{a_n\}_{n\in\mathbb{N}} \in l^p$.

2. Show that when $p = 1$, there exists $b = \{b_n\}_{n\in\mathbb{N}}$ such that

$$\|b\|_{l^\infty} := \max_{n\in\mathbb{N}} |b_n|$$

is finite for which the formula (2) holds for every $a = \{a_n\}_{n\in\mathbb{N}} \in l^1$.

Real analysis qualifying examination

National Taiwan Normal University

Autumn 2024

Guidelines. To use any theorem formulate that theorem prior to your proof and indicate clearly when and how you apply the theorem in your proof. In particular, all results—excluding the exercises—from [WZ77] may be employed as precedingly indicated. Every question is worth 20 credits.

Question 1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable. Prove that there exists a Borel measurable function $g : \mathbf{R} \to \mathbf{R}$ such that $f = g$ Lebesgue almost everywhere in $\mathbf{R}$.

Question 2. Prove or disprove the following statement.

Suppose that $f_i : [0, 1] \to [0, 1]$, corresponding to positive integers i, form a sequence of Lebesgue measurable functions, $g : [0, 1] \to [0, 1]$ is a Lebesgue measurable function, and for every subsequence $j(1) < j(2) < j(3) < \cdots$, there exists a subsequence $k(1) < k(2) < k(3) < \cdots$ such that

$$\lim_{l\to\infty} f_{j(k(l))} = g \quad \text{Lebesgue almost everywhere in } [0, 1].$$

Then, there holds

$$\lim_{i\to\infty} f_i = g \quad \text{Lebesgue almost everywhere in } [0, 1].$$

(By a subsequence we mean a strictly increasing sequence of positive integers.)

Question 3. Prove or disprove the following statement.

Suppose that $f_i : [0, 1] \to [0, 1]$, corresponding to positive integers i, form a sequence of Lebesgue measurable functions, $g : [0, 1] \to [0, 1]$ is a Lebesgue measurable function, and for every subsequence $j(1) < j(2) < j(3) < \cdots$, there exists a subsequence $k(1) < k(2) < k(3) < \cdots$ such that

$$\lim_{l\to\infty} f_{j(k(l))} = g \quad \text{in Lebesgue measure on } [0, 1].$$

Then, there holds

$$\lim_{i\to\infty} f_i = g \quad \text{in Lebesgue measure on } [0, 1].$$

Question 4. We identify $\mathbf{R}^2 \simeq \mathbf{R} \times \mathbf{R}$.

Suppose $f_i : \mathbf{R}^2 \to \mathbf{R}$, corresponding to positive integers i, form a sequence of nonnegative Lebesgue measurable functions and

$$\lim_{i\to\infty} \iint_{\mathbf{R}^2} f_i(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Prove that there exists a subsequence $j(1) < j(2) < j(3) < \cdots$ such that, for Lebesgue almost all y in $\mathbf{R}$,

$$\lim_{k\to\infty} \int_{\mathbf{R}} f_{j(k)}(x, y)\, \mathrm{d}x = 0.$$

Question 5. Suppose S is a countably infinite subset of $\mathbf{R}$, $g : S \to \mathbf{R}$, $g(s) > 0$ for $s \in S$, and, for some one-to-one enumeration $s_1, s_2, s_3, \ldots$ of S, there holds

$$\sum_{i=1}^{\infty} g(s_i) = 1.$$

Prove that there exists a function $f : \mathbf{R} \to \mathbf{R}$ with the following three properties.

(1) Whenever $x, y \in \mathbf{R}$ and $x \le y$, we have $f(x) \le f(y)$.

(2) We have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$.

(3) For $s \in S$, there holds $f(s+) - f(s-) = g(s)$.

(For $x \in \mathbf{R}$, the limit of f at x from the right is denoted by $f(x+)$ and the limit of f at x from the left by $f(x-)$.)

References

[WZ77] Richard L. Wheeden and Antoni Zygmund. *Measure and integral*, volume Vol. 43 of *Pure and Applied Mathematics*. Marcel Dekker, Inc., New York-Basel, 1977. An introduction to real analysis.