Total score (100 points) ：

1 *(10 points) Find the general solution of $x' = Ax$ with*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}.$$

Here, eigenvalues of the matrix A are 1, 1, and 0.

2 *(a) (5 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\frac{dx}{dt} = y$$
$$\frac{dt}{dt} = -x - \mu(x^2 - 1)y$$

(b) (15 points) Find the stabilities of the critical point $(0, 0)$ for the cases $\mu > 0$ and $\mu < 0$.

3 *(15 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

where $a, b, c > 0$. Let $y = x'$. Show that $(0, 0)$ is Liapunov stable using Liapunov function of the form $V(x, y) = \alpha x^2 + \beta x^4 + \gamma y^2$ for $\alpha, \beta, \gamma > 0$.

4 *(15 points) Consider the following system*

$$\frac{dx}{dt} = x - y - x^3$$
$$\frac{dt}{dt} = x + y - y^3$$

Show that there is at least one stable limit cycle in the region $A = \{(x, y) \in \mathbb{R}^2 | 1 \leq |(x, y)| \leq \sqrt{2}\}$.

5 *(15 points) If* $C \geq 0$ *and* $u, v : [0, \beta] \to [0, \infty)$ *are continuous and*

$$u(t) \leq C + \int_0^t u(s)v(s)ds$$

for all $t \in [0, \beta]$*, then*

$$u(t) \leq Ce^{v(t)},$$

where $v(t) = \int_0^t v(s)ds$.

6 *(a) (15 points) Let* $n = 2$. *For any* 2×2 *constant real matrix* A*, show that there exists an invertible real matrix* P *such that the matrix*

$$B = P^{-1}AP$$

has one of the following forms

$$\text{(i)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{(ii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{(iii)} \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $\lambda, \mu, a, b \in \mathbb{R}$. *Find* P *explicitly.*

(b) (10 points) Let $A = \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix}$. *where* $\lambda, \mu, \alpha \in \mathbb{R}$. *Solve the initial value problem:* $x' = Ax,\ x(0) = x_0$.

Total score (100 points) ：

1 *(20 points) Find the fundamental matrix of $x' = Ax$ with*

$$A = \begin{pmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{pmatrix}.$$

2 *(a) (4 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\frac{dx}{dt} = y$$
$$\frac{dt}{dt} = -x - \mu(x^2-1)y$$

(b) (16 points) Characterize the types and the stabilities of the critical point $(0,0)$ for the cases $\mu > 0$ and $\mu < 0$.

3 *(20 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

where $a, b, c > 0$. Let $y = x'$. Show that $(0,0)$ is Liapunov stable using Liapunov function of the form $V(x,y) = \alpha x^2 + \beta x^4 + \gamma y^2$ for $\alpha, \beta, \gamma > 0$.

4 *Consider the following initial value problem (IVP)*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \tag{2}$$

(i)(6 points) Show that the solution of the IVP$(1)(2)$ stays in the region $D = \{(x,y) | \frac{1}{2} \le \sqrt{x^2+y^2} \le 2\}$ whenever it exists.

(ii)(6 points) Show that there exists a unique solution for IVP$(1)(2)$, which exists for all $t \in \mathbb{R}$.

(iii)(8 points) Find the equilibrium of (1). Discuss the asymptotical behavior of the solution for IVP$(1)(2)$ as $t \to \infty$.

5 *(20 points) Let* $r, k,$ *and* f *be real and continuous functions which satisfy* $r(t) \geq 0$*,* $k(t) \geq 0$*, and*

$$r(t) \leq f(t) + \int_a^t k(s) r(s) ds, \quad a \leq t \leq b.$$

Show that

$$r(t) \leq f(t) + \int_a^t f(s) k(s) \exp\left[\int_s^t k(u) du\right] ds, \quad a \leq t \leq b.$$

Total score (100 points) :

1 *(20 points) Solve the following nonhomogeneous system of differential equations:*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ t \end{pmatrix},$$

where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ *is a vector-valued function of* t*. Find the general solution for* $\mathbf{x}(t)$*.*

2 *(20 points) Assume that* $p(t)$ *and* $q(t)$ *are continuous functions on an interval* I*. Let* $y_1(t)$ *and* $y_2(t)$ *be a fundamental set of solutions to the second-order linear homogeneous differential equation*

$$y'' + p(t)y' + q(t)y = 0.$$

(a) *The Wronskian of* $y_1(t)$ *and* $y_2(t)$ *is defined by*

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}.$$

Prove that

$$W[y_1, y_2](t) = c\, e^{-\int p(t)\, dt},$$

where c *is a constant.*

(b) *Show that between any two consecutive zeros of* $y_1(t)$*, there is exactly one zero of* $y_2(t)$*. (Hint: Consider the function* $y_1(t)/y_2(t)$ *and use proof by contradiction.)*

3 *(20 points) Consider the boundary value problem*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad \alpha y'(1) + y(1) = 0,$$

where α *is a given constant.*

(a) *Show that, for all values of* α*, there exists an infinite sequence of positive eigenvalues* λ*.*

(b) *Suppose* $-1 < \alpha < 0$*. Show that there exists exactly one negative eigenvalue, and that this eigenvalue increases as* α *decreases.*

4 *(20 points) Consider the system given by*

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy, \\ \dfrac{dy}{dt} = cxy - dy, \end{cases} \qquad \text{with } a, b, c, d > 0, \quad x(0) = x_0 > 0, \ y(0) = y_0 > 0.$$

(a) *Find all equilibrium points of the system. Identify which equilibrium lies in the positive quadrant of the phase plane.*

(b) *Derive the equation for the trajectories in the phase plane by computing* $\frac{dy}{dx}$*. Show that the system admits a conserved quantity (first integral) of the form*

$$\frac{y - y^*}{y} dy + \frac{c}{b} \frac{x - x^*}{x} dx = 0,$$

where $x^* = \frac{d}{c}$, $y^* = \frac{a}{b}$.

(c) *Define the function*

$$V(x, y) = \left[y - y^* - y^* \ln\left(\frac{y}{y^*}\right) \right] + \frac{c}{b} \left[x - x^* - x^* \ln\left(\frac{x}{x^*}\right) \right].$$

Prove that $\frac{dV}{dt} = 0$ *along solutions of the system. What does this imply about the behavior of the system?*

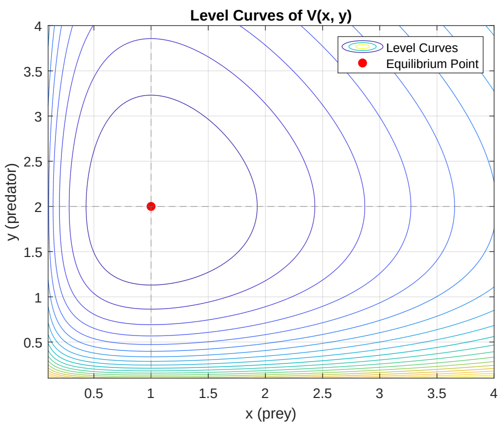

Figure 1: Contour plot (level curves) of the Lyapunov function $V(x, y)$ for the predator-prey system.

(d) *Explain why the interior equilibrium point* (x^*, y^*) *is Lyapunov stable but not asymptotically stable. Use a phase portrait sketch or qualitative description to support your answer.*

5 *(20 points)*

(a) *State the Bendixson Criterion. Explain the condition under which a two-dimensional autonomous system does not admit any periodic orbits within a simply connected domain.*

(b) *Let* $h(x, y) \in C^1(D)$*, where* $D \subset \mathbb{R}^2$ *is a simply connected domain. Assume that the expression*

$$\frac{\partial(fh)}{\partial x} + \frac{\partial(gh)}{\partial y}$$

is of one sign and never zero throughout D*. Prove that the planar autonomous system*

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

admits no periodic orbits entirely contained in D*. This is known as Dulac's Criterion.*

(c) *Consider the model*

$$\begin{cases} \dfrac{dx}{dt} = \gamma_1 x \left(1 - \dfrac{x}{K_1}\right) - \alpha xy, \\ \dfrac{dy}{dt} = \gamma_2 y \left(1 - \dfrac{y}{K_2}\right) - \beta xy, \end{cases}$$

where all parameters $\gamma_1, \gamma_2, \alpha, \beta, K_1, K_2$ *are positive constants. Use part (b) to show that the system admits no periodic orbits entirely contained in the first quadrant. Hint:* $h(x, y) = \frac{1}{xy}$.