Combinatorics Qualifying Examination

NTNU Math Ph.D. Program, Fall 2019

1. (10%) What is the expected number of fixed points of a permutation in $S(n)$?

2. (10%) Let a_n be the number of n-words over the alphabet $\{0, 1, 2\}$ that contain no neighboring 0's, e.g., $a_1 = 3$, $a_2 = 8$, $a_3 = 22$. Find the generating function of a_n.

3. (15%) Let a_n be the number of self-conjugate partitions of n. Prove the following identities:

   (a) $\displaystyle\sum_{n\geq 0} a_n z^n = \prod_{i\geq 1}(1 + z^{2i-1})$.

   (b) $\displaystyle\sum_{n\geq 0} \frac{q^n z^{n^2}}{(1 - z^2)(1 - z^4)\cdots(1 - z^{2n})} = \prod_{i\geq 1}(1 + qz^{2i-1})$

   (c) $\displaystyle\prod_{i\geq 1}(1 + z^i) = \prod_{i\geq 1}(1 - z^{2i-1})^{-1}$

4. Let $i_n^{(r)}$ be the number of permutations in $S(n)$ with no cycles of length greater than r.

   (a) (5%) Prove $i_{n+1}^{(2)} = i_n^{(2)} + n i_{n-1}^{(2)}$.

   (b) (10%) Prove $\displaystyle i_{n+1}^{(r)} = \sum_{k=n-r+1}^{n} n^{\underline{n-k}} i_k^{(r)}$.

5. (10%) A permutation $\sigma \in S(n)$ is called connected if for any k, $1 \leq k < n$, $\{\sigma(1), \sigma(2), \ldots, \sigma(k)\} \neq [k]$. Find the number of connected permutations in $S(8)$.

6. (10%) Toss a fair coin until you get heads for the n-th time. Let X be the number of throws necessary. What are $P_X(z)$, $E(X)$, and $Var(X)$?

7. (10%) Let a_n be the number of ordered set partitions of $\{1, \ldots, n\}$. Compute $\displaystyle\sum_{n\geq 0} a_n \frac{z^n}{n!}$.

8. (10%) Let S be the family of k-subsets of $\{1, 2, \ldots, 2n\}$. For $A \in S$ let $w(A) = \displaystyle\sum_{i\in A} i$, and set $S^+ = \{A \in S \mid w(A) \text{ even}\}$, $S^- = \{A \in S \mid w(A) \text{ odd}\}$. Find an alternating involution to show that
$$|S^+| - |S^-| = \begin{cases} 0, & k \text{ odd;} \\ (-1)^{k/2}\binom{n}{k/2}, & k \text{ even.} \end{cases}$$

9. (10%) Show that any permutation of $\{1, 2, \ldots, mn + 1\}$ contains an increasing subword of length $m + 1$ or a decreasing subword of length $n + 1$.

Combinatorics Qualifying Examination

NTNU Math Ph.D. Program, Spring 2023

(10 points each)

1. (a) Evaluate
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}.$$
   (b) Evaluate
$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}.$$

2. It is known there exists a unique sequence a_n of real numbers ($n \geq 0$) such that for each n we have
$$\sum_{k=0}^{n} a_k a_{n-k} = 1.$$
   (a) Find the generating function of a_n
   (b) Find a_n

3. Denote $x^{\underline{n}} = x(x-1)\dots(x-n+1)$ and $x^{\overline{n}} = x(x+1)\dots(x+n-1)$. $S_{n,k}$ is the Stirling number of the second kind. $s_{n,k}$ is the (signless) Stirling number of the first kind.
   (a) Prove that
$$x^n = \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}.$$
   (b) Prove that
$$x^{\overline{n}} = \sum_{k=0}^{n} s_{n,k} x^k.$$

4. State and prove the q-binomial theorem.

5. Let V be an n-dimensional vector space over the finite field $GF(q)$, where q is a prime power. Prove that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ equals the number of k-dimensional subspaces of V.

6. Count the number of plane partitions whose number of rows is no greater than 3, number of columns is no greater than 3, and height are no greater than 4.

7. Color the vertices of a cube in 3 colors x, y, z. The cube is acted by its symmetries. Two colorings are equivalent if one can be obtained by applying a symmetry. An equivalent class is called a pattern. Compute the generating function of the pattern polynomials for all possible patterns.

8. Let $a \leq b$ are two elements of a poset P. δ is the identity and ζ is the zeta function of the incidence algebra of P. Define $\eta := \zeta - \delta$.

   (a) Show that
   $$\sum_{k\geq 0} \eta^k(a,b) = \frac{1}{2\delta - \zeta}(a,b)$$

   (b) What happens if appy this to the Boolean algebra $\mathbb{B}(n)$?

9. Prove
$$\prod_{k\geq 1}(1-q^{4k-3})(1-q^{4k-1})(1-q^{4k}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}.$$

10. How many rooted forests are there on $\{1, ..., n\}$ with k components?

Combinatorics Qualifying Examination

NTNU Math Ph.D. Program, Fall 2023

1. (10%) In a 6×6 grid, how many ways are there to select 30 squares such that there are no consecutive 6 squares in any row, column, or diagonal?

2. (10%) Color the vertices of a floating cube in 3 colors. Compute the generating function of the pattern polynomials for all possible patterns(pattern inventory).

3. (10%) Let a_n be the number of n-words over the alphabet $\{0, 1, 2\}$ that contain no neighboring 0's, e.g., $a_1 = 3$, $a_2 = 8$, $a_3 = 22$. Find the (ordinary) generating function of a_n.

4. (10%) Let a_n be the number of ordered set partitions of $\{1, \ldots, n\}$. Compute $\sum_{n \geq 0} a_n \frac{z^n}{n!}$.

5. (10%) $S_{n,k}$ is the Stirling number of the second kind. Prove that $x^n = \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}$.

6. (15%) Prove the following identities:

   (a) $\prod_{i \geq 1}(1 + z^i) = \prod_{i \geq 1}(1 - z^{2i-1})^{-1}$

   (b) $\sum_{n \geq 0} \frac{q^n z^{n^2}}{(1 - z^2)(1 - z^4) \cdots (1 - z^{2n})} = \prod_{i \geq 1}(1 + q z^{2i-1})$

   (c) $\prod_{i \geq 1}(1 + z q^i)(1 + z^{-1} q^{i-1})(1 - q^i) = \sum_{i=-\infty}^{\infty} q^{\frac{i(i+1)}{2}} z^i$

7. (10%) Show that any permutation of $\{1, 2, \ldots, mn + 1\}$ contains an increasing subword of length $m + 1$ or a decreasing subword of length $n + 1$.

8. (15%)
   (a) How many unrooted spanning trees are there in the labeled complete graph K_n?

   (b) How many rooted spanning forests with k components are there in the labeled complete graph K_n?

   (c) How many unrooted spanning trees are there in the labeled complete bipartite graph $K_{m,n}$?

9. (10%) Let λ be a partition with Ferrers diagram D. For each cell s, a number will be filled in. This number represents the number of ways to go from the lowest cell below s to the cell farthest to the right of s. Let M be the Durfee square of λ (largest square contained in D).

   Prove $\det M = 1$. Example: $\lambda = 43311$, then $\begin{vmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1$.

6	3	1	1
3	2	1	
1	1	1	
1			
1			

Combinatorics Qualifying Examination

NTNU Math Ph.D. Program, Spring 2025

1. (10%) Find the expected number of fixed points of a permutation in $\mathfrak{S}_n$.

2. (10%) Consider $\lambda = 3221 \in \mathrm{Par}(8)$. Determine the number of standard Young tableaux of shape λ, and the number of semistandard Young tableaux of shape λ.

3. (10%) Determine and prove a formula for the number of parking sequences of length n.

4. (10%) In how many permutations of $1, 2, \ldots, n$ does every element (except the first) appear only after at least one of its neighbors, $k-1$ or $k+1$, has already appeared? For example: $324516, 435216$, but not 351246.

5. (10%) Prove the identity:

$$\prod_{i\geq 1} \frac{1}{1-qz^i} = \sum_{k\geq 0} \frac{q^k z^k}{\prod_{i=1}^{k}(1-z^i)} = \sum_{k\geq 0} \frac{q^k z^{k^2}}{\prod_{i=1}^{k}(1-z^i)(1-qz^i)}$$

6. (10%) Prove that the exponential generating function for the number of permutations in $\mathfrak{S}_n$ in which every cycle has odd length is $\sqrt{\frac{1+z}{1-z}}$.

7. (10%) Let $f(n,k)$ be the number of permutations in $\mathfrak{S}_n$ which have no cycle of length k. Compute $\lim_{n\to\infty} f(n,k)/n!$.

8. (10%) Show that any permutation in $\mathfrak{S}_{mn+1}$ contains an increasing subword of length $m+1$ or a decreasing subword of length $n+1$.

9. (20%) Let $w \in \mathfrak{S}_n$. For any finite set S of positive integers, set $x^S = \prod_{i\in S} x_i$.

   (a) Let $R(w)$ be the set of positions of the left-to-right maxima of w. For instance, $R(3265174) = \{1, 3, 6\}$ and $x^{R(3265174)} = x_1 x_3 x_6$. Show that

   $$\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} x^{R(w)} = x_1(x_2+q)(x_3+q+q^2)\cdots(x_n+q+q^2+\cdots+q^{n-1}).$$

   (b) Let $V(w)$ be the set of the records themselves. For instance, $V(3265174) = \{3, 6, 7\}$. Show that

   $$\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} x^{V(w)} = (x_1+q+q^2+\cdots+q^{n-1})(x_2+q+q^2+\cdots+q^{n-2})\cdots(x_{n-1}+\mathrm{q}\,)x_n.$$