Total score (100 points) ：

1 *(10 points) Find the general solution of* $x' = Ax$ *with*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}.$$

Here, eigenvalues of the matrix A *are* 1, 1, *and* 0.

2 *(a) (5 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\frac{dx}{dt} = y$$
$$\frac{dt}{dt} = -x - \mu(x^2-1)y$$

(b) (15 points) Find the stabilities of the critical point $(0,0)$ *for the cases* $\mu > 0$ *and* $\mu < 0$.

3 *(15 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

where $a, b, c > 0$. *Let* $y = x'$. *Show that* $(0,0)$ *is Liapunov stable using Liapunov function of the form* $V(x,y) = \alpha x^2 + \beta x^4 + \gamma y^2$ *for* $\alpha, \beta, \gamma > 0$.

4 *(15 points) Consider the following system*

$$\frac{dx}{dt} = x - y - x^3$$
$$\frac{dt}{dt} = x + y - y^3$$

Show that there is at least one stable limit cycle in the region $A = \{(x,y) \in \mathbb{R}^2 | 1 \le |(x,y)| \le \sqrt{2}\}$.

5 *(15 points) If* $C \geq 0$ *and* $u, v : [0, \beta] \to [0, \infty)$ *are continuous and*

$$u(t) \leq C + \int_0^t u(s)v(s)ds$$

for all $t \in [0, \beta]$, *then*

$$u(t) \leq Ce^{v(t)},$$

where $v(t) = \int_0^t v(s)ds$.

6 *(a) (15 points) Let* $n = 2$. *For any* 2×2 *constant real matrix* A, *show that there exists an invertible real matrix* P *such that the matrix*

$$B = P^{-1}AP$$

has one of the following forms

$$\text{(i)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{(ii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{(iii)} \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $\lambda, \mu, a, b \in \mathbb{R}$. *Find* P *explicitly.*

(b) (10 points) Let $A = \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix}$. *where* $\lambda, \mu, \alpha \in \mathbb{R}$. *Solve the initial value problem:* $x' = Ax,\ x(0) = x_0$.

Total score (100 points) ：

1 *(20 points) Find the fundamental matrix of* $x' = Ax$ *with*

$$A = \begin{pmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{pmatrix}.$$

2 *(a) (4 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dt}{dt} &= -x - \mu(x^2-1)y \end{aligned}$$

(b) (16 points) Characterize the types and the stabilities of the critical point $(0,0)$ *for the cases* $\mu > 0$ *and* $\mu < 0$.

3 *(20 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

where $a, b, c > 0$. *Let* $y = x'$. *Show that* $(0,0)$ *is Liapunov stable using Liapunov function of the form* $V(x,y) = \alpha x^2 + \beta x^4 + \gamma y^2$ *for* $\alpha, \beta, \gamma > 0$.

4 *Consider the following initial value problem (IVP)*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y - x(x^2+y^2) \\ -x + y - y(x^2+y^2) \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \tag{2}$$

(i)(6 points) Show that the solution of the IVP$(1)(2)$ *stays in the region* $D = \{(x,y) |\ \frac{1}{2} \le \sqrt{x^2+y^2} \le 2\}$ *whenever it exists.*

(ii)(6 points) Show that there exists a unique solution for IVP$(1)(2)$, *which exists for all* $t \in \mathbb{R}$.

(iii)(8 points) Find the equilibrium of (1). *Discuss the asymptotical behavior of the solution for IVP*$(1)(2)$ *as* $t \to \infty$.

5 *(20 points) Let* $r, k,$ *and* f *be real and continuous functions which satisfy* $r(t) \geq 0$, $k(t) \geq 0$, *and*

$$r(t) \leq f(t) + \int_a^t k(s)r(s)ds, \quad a \leq t \leq b.$$

Show that

$$r(t) \leq f(t) + \int_a^t f(s)k(s) \exp\left[\int_s^t k(u)du\right] ds, \quad a \leq t \leq b.$$

Total score (100 points) :

1 *(20 points) Solve the following nonhomogeneous system of differential equations:*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ t \end{pmatrix},$$

where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ *is a vector-valued function of* t. *Find the general solution for* $\mathbf{x}(t)$.

2 *(20 points) Assume that* $p(t)$ *and* $q(t)$ *are continuous functions on an interval* I. *Let* $y_1(t)$ *and* $y_2(t)$ *be a fundamental set of solutions to the second-order linear homogeneous differential equation*

$$y'' + p(t)y' + q(t)y = 0.$$

(a) *The Wronskian of* $y_1(t)$ *and* $y_2(t)$ *is defined by*

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}.$$

Prove that

$$W[y_1, y_2](t) = c\, e^{-\int p(t)\, dt},$$

where c *is a constant.*

(b) *Show that between any two consecutive zeros of* $y_1(t)$, *there is exactly one zero of* $y_2(t)$. *(Hint: Consider the function* $y_1(t)/y_2(t)$ *and use proof by contradiction.)*

3 *(20 points) Consider the boundary value problem*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad \alpha y'(1) + y(1) = 0,$$

where α *is a given constant.*

(a) *Show that, for all values of* α, *there exists an infinite sequence of positive eigenvalues* λ.

(b) *Suppose* $-1 < \alpha < 0$. *Show that there exists exactly one negative eigenvalue, and that this eigenvalue increases as* α *decreases.*

4 *(20 points) Consider the system given by*

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy, \\ \dfrac{dy}{dt} = cxy - dy, \end{cases} \qquad \text{with } a, b, c, d > 0, \quad x(0) = x_0 > 0, \; y(0) = y_0 > 0.$$

(a) *Find all equilibrium points of the system. Identify which equilibrium lies in the positive quadrant of the phase plane.*

(b) *Derive the equation for the trajectories in the phase plane by computing* $\frac{dy}{dx}$*. Show that the system admits a conserved quantity (first integral) of the form*

$$\frac{y - y^*}{y} dy + \frac{c}{b} \frac{x - x^*}{x} dx = 0,$$

where $x^* = \frac{d}{c}$, $y^* = \frac{a}{b}$.

(c) *Define the function*

$$V(x, y) = \left[y - y^* - y^* \ln\left(\frac{y}{y^*}\right) \right] + \frac{c}{b} \left[x - x^* - x^* \ln\left(\frac{x}{x^*}\right) \right].$$

Prove that $\frac{dV}{dt} = 0$ *along solutions of the system. What does this imply about the behavior of the system?*

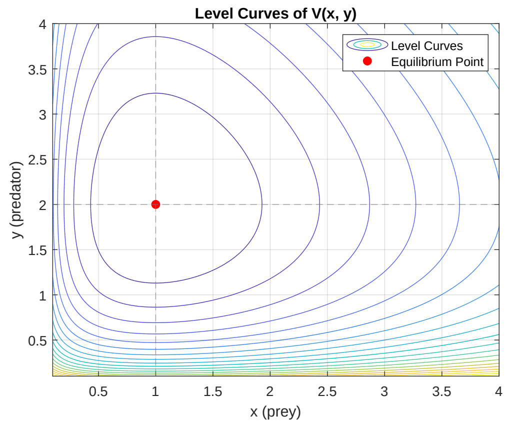

Figure 1: Contour plot (level curves) of the Lyapunov function $V(x, y)$ for the predator-prey system.

(d) Explain why the interior equilibrium point (x^*, y^*) is Lyapunov stable but not asymptotically stable. Use a phase portrait sketch or qualitative description to support your answer.

5 *(20 points)*

(a) State the Bendixson Criterion. Explain the condition under which a two-dimensional autonomous system does not admit any periodic orbits within a simply connected domain.

(b) Let $h(x, y) \in C^1(D)$, where $D \subset \mathbb{R}^2$ is a simply connected domain. Assume that the expression

$$\frac{\partial(fh)}{\partial x} + \frac{\partial(gh)}{\partial y}$$

is of one sign and never zero throughout D. Prove that the planar autonomous system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

admits no periodic orbits entirely contained in D. This is known as Dulac's Criterion.

(c) Consider the model

$$\begin{cases} \dfrac{dx}{dt} = \gamma_1 x \left(1 - \dfrac{x}{K_1}\right) - \alpha xy, \\ \dfrac{dy}{dt} = \gamma_2 y \left(1 - \dfrac{y}{K_2}\right) - \beta xy, \end{cases}$$

where all parameters $\gamma_1, \gamma_2, \alpha, \beta, K_1, K_2$ are positive constants. Use part (b) to show that the system admits no periodic orbits entirely contained in the first quadrant. Hint: $h(x, y) = \frac{1}{xy}$.

Ph.D. Qualifying Examination in ODE 2025

Each Problem: 20 points

1. Consider the following ordinary differential equations:

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where f is a continuous function on the domain $D := \{(t,x)||t-t_0| \le a, |x-x_0| \le b\} \subset (\mathbf{R} \times \mathbf{R^n})$.

   (i) Prove that: there exists $0 < \delta < a$ such that equation (1) has a solution $x(t)$ $\forall$ $t_0 - \delta < t < t_0 + \delta$.

   (ii) Prove that: if $f(t,x)$ is a Lipschitz continuous function in x, then the solution of equation (1) is unique.

2. Consider the following system of ODE:

$$X'(t) = AX(t), \quad \text{where} \quad A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}. \tag{2}$$

   (i) Find $e^{At} := I + \Sigma_{k=1}^{\infty} \frac{A^k t^k}{k!}$.

   (ii) Find the general solutions of equation (2).

3. (i) Assume that q_1 and q_2 are continuous functions satisfying $q_1(t) \le q_2(t)$ on an interval $I \subset \mathbf{R}$. Let ϕ_1 and ϕ_2 be non-trivial solutions of equations

$$y'' + q_1(t)y = 0$$

and

$$y'' + q_2(t)y = 0$$

respectively, on I. Prove that: between any two consecutive zeroes α and β of ϕ_1, there exists at least one zero of ϕ_2 unless $\phi_1(t) \equiv \phi_2(t)$ on (α, β).

   (ii) Applying the result of above (i) to show that the equation

$$y''(x) + xy(x) = 0$$

has a non-trivial solution with infinitely many zeros on the inteerval $(1, \infty)$.

4. Consider the following two dimensional ODE

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy, \end{cases} \tag{3}$$

where a, b, c, d are real parameters. Discuss all the stability cases of equilibrium $(0,0)$ depending on these parameters.

5. Consider the nonlinear system:

$$\begin{cases} x' = y - x^3, \\ y' = -x - y^3. \end{cases} \tag{4}$$

Prove that: $(0,0)$ is the only equilibrium of (4), and $(0,0)$ is globally asymptotically stable. (**Hint** You can use the Lyapunov function $V(x,y) = x^2 + y^2$.)