Total score (100 points) ：

1 *(10 points) Find the general solution of* $x' = Ax$ *with*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 3 & -4 \\ 1 & 2 & -1 \end{pmatrix}.$$

Here, eigenvalues of the matrix A *are* 1, 1, *and* 0.

2 *(a) (5 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dt}{dt} &= -x - \mu(x^2 - 1)y \end{aligned}$$

(b) (15 points) Find the stabilities of the critical point $(0, 0)$ *for the cases* $\mu > 0$ *and* $\mu < 0$.

3 *(15 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

where $a, b, c > 0$. *Let* $y = x'$. *Show that* $(0, 0)$ *is Liapunov stable using Liapunov function of the form* $V(x, y) = \alpha x^2 + \beta x^4 + \gamma y^2$ *for* $\alpha, \beta, \gamma > 0$.

4 *(15 points) Consider the following system*

$$\begin{aligned} \frac{dx}{dt} &= x - y - x^3 \\ \frac{dt}{dt} &= x + y - y^3 \end{aligned}$$

Show that there is at least one stable limit cycle in the region $A = \{(x, y) \in \mathbb{R}^2 | 1 \leq |(x, y)| \leq \sqrt{2}\}$.

5 *(15 points) If $C \geq 0$ and $u, v : [0, \beta] \to [0, \infty)$ are continuous and*

$$u(t) \leq C + \int_0^t u(s)v(s)ds$$

for all $t \in [0, \beta]$, then

$$u(t) \leq Ce^{v(t)},$$

where $v(t) = \int_0^t v(s)ds$.

6 *(a) (15 points) Let $n = 2$. For any 2×2 constant real matrix A, show that there exists an invertible real matrix P such that the matrix*

$$B = P^{-1}AP$$

has one of the following forms

$$\textit{(i)} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \textit{(ii)} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \textit{(iii)} \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $\lambda, \mu, a, b \in \mathbb{R}$. Find P explicitly.

(b) (10 points) Let $A = \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix}$. where $\lambda, \mu, \alpha \in \mathbb{R}$. Solve the initial value problem: $x' = Ax,\ x(0) = x_0$.

Total score (100 points) ：

1 *(20 points) Find the fundamental matrix of* $x' = Ax$ *with*

$$A = \begin{pmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{pmatrix}.$$

2 *(a) (4 points) Show that the van der Pol equation*

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dt}{dt} &= -x - \mu(x^2 - 1)y \end{aligned}$$

(b) (16 points) Characterize the types and the stabilities of the critical point $(0, 0)$ *for the cases* $\mu > 0$ *and* $\mu < 0$.

3 *(20 points) Consider the nonlinear oscillator*

$$x'' + cx' + ax + bx^3 = 0,$$

where $a, b, c > 0$. *Let* $y = x'$. *Show that* $(0, 0)$ *is Liapunov stable using Liapunov function of the form* $V(x, y) = \alpha x^2 + \beta x^4 + \gamma y^2$ *for* $\alpha, \beta, \gamma > 0$.

4 *Consider the following initial value problem (IVP)*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y - x(x^2 + y^2) \\ -x + y - y(x^2 + y^2) \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \tag{2}$$

(i)(6 points) Show that the solution of the IVP(1)(2) *stays in the region* $D = \{(x, y) | \frac{1}{2} \leq \sqrt{x^2 + y^2} \leq 2\}$ *whenever it exists.*

(ii)(6 points) Show that there exists a unique solution for IVP(1)(2), *which exists for all* $t \in \mathbb{R}$.

(iii)(8 points) Find the equilibrium of (1). *Discuss the asymptotical behavior of the solution for IVP*(1)(2) *as* $t \to \infty$.

5 *(20 points) Let r, k, and f be real and continuous functions which satisfy $r(t) \geq 0$, $k(t) \geq 0$, and*

$$r(t) \leq f(t) + \int_a^t k(s)r(s)ds, \quad a \leq t \leq b.$$

Show that

$$r(t) \leq f(t) + \int_a^t f(s)k(s) \exp\left[\int_s^t k(u)du\right]ds, \quad a \leq t \leq b.$$

Total score (100 points) :

1 *(20 points) Solve the following nonhomogeneous system of differential equations:*

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ t \end{pmatrix},$$

where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ *is a vector-valued function of* t*. Find the general solution for* $\mathbf{x}(t)$*.*

2 *(20 points) Assume that* $p(t)$ *and* $q(t)$ *are continuous functions on an interval* I*. Let* $y_1(t)$ *and* $y_2(t)$ *be a fundamental set of solutions to the second-order linear homogeneous differential equation*

$$y'' + p(t)y' + q(t)y = 0.$$

(a) The Wronskian of $y_1(t)$ *and* $y_2(t)$ *is defined by*

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}.$$

Prove that

$$W[y_1, y_2](t) = c\, e^{-\int p(t)\, dt},$$

where c *is a constant.*

(b) Show that between any two consecutive zeros of $y_1(t)$*, there is exactly one zero of* $y_2(t)$*. (Hint: Consider the function* $y_1(t)/y_2(t)$ *and use proof by contradiction.)*

3 *(20 points) Consider the boundary value problem*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad \alpha y'(1) + y(1) = 0,$$

where α *is a given constant.*

(a) Show that, for all values of α*, there exists an infinite sequence of positive eigenvalues* λ*.*

(b) Suppose $-1 < \alpha < 0$*. Show that there exists exactly one negative eigenvalue, and that this eigenvalue increases as* α *decreases.*

4 *(20 points) Consider the system given by*

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy, \\ \dfrac{dy}{dt} = cxy - dy, \end{cases} \qquad \text{with } a, b, c, d > 0, \quad x(0) = x_0 > 0,\ y(0) = y_0 > 0.$$

(a) Find all equilibrium points of the system. Identify which equilibrium lies in the positive quadrant of the phase plane.

(b) Derive the equation for the trajectories in the phase plane by computing $\frac{dy}{dx}$*. Show that the system admits a conserved quantity (first integral) of the form*

$$\frac{y - y^*}{y} dy + \frac{c}{b} \frac{x - x^*}{x} dx = 0,$$

where $x^* = \frac{d}{c}$, $y^* = \frac{a}{b}$.

(c) Define the function

$$V(x, y) = \left[y - y^* - y^* \ln\left(\frac{y}{y^*}\right) \right] + \frac{c}{b} \left[x - x^* - x^* \ln\left(\frac{x}{x^*}\right) \right].$$

Prove that $\frac{dV}{dt} = 0$ *along solutions of the system. What does this imply about the behavior of the system?*

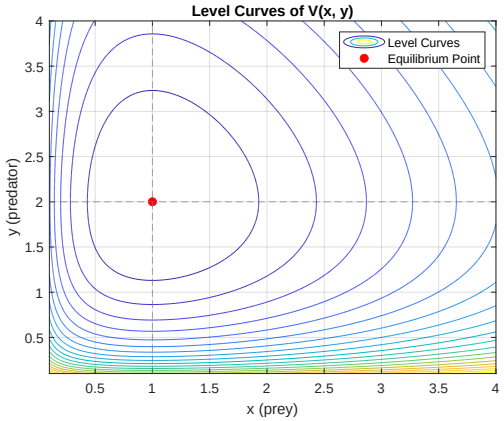


Figure 1: Contour plot (level curves) of the Lyapunov function $V(x, y)$ for the predator-prey system.

(d) Explain why the interior equilibrium point (x^*, y^*) is Lyapunov stable but not asymptotically stable. Use a phase portrait sketch or qualitative description to support your answer.

5 *(20 points)*

(a) State the Bendixson Criterion. Explain the condition under which a two-dimensional autonomous system does not admit any periodic orbits within a simply connected domain.

(b) Let $h(x, y) \in C^1(D)$, where $D \subset \mathbb{R}^2$ is a simply connected domain. Assume that the expression

$$\frac{\partial(fh)}{\partial x} + \frac{\partial(gh)}{\partial y}$$

is of one sign and never zero throughout D. Prove that the planar autonomous system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

admits no periodic orbits entirely contained in D. This is known as Dulac's Criterion.

(c) Consider the model

$$\begin{cases} \dfrac{dx}{dt} = \gamma_1 x \left(1 - \dfrac{x}{K_1}\right) - \alpha xy, \\ \dfrac{dy}{dt} = \gamma_2 y \left(1 - \dfrac{y}{K_2}\right) - \beta xy, \end{cases}$$

where all parameters $\gamma_1, \gamma_2, \alpha, \beta, K_1, K_2$ are positive constants. Use part (b) to show that the system admits no periodic orbits entirely contained in the first quadrant. Hint: $h(x, y) = \frac{1}{xy}$.

Ph.D. Qualifying Examination in ODE 2025

Each Problem: 20 points

1. Consider the following ordinary differential equations:

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where f is a continuous function on the domain $D := \{(t,x)||t-t_0| \leq a, |x-x_0| \leq b\} \subset (\mathbf{R} \times \mathbf{R^n})$.

   (i) Prove that: there exists $0 < \delta < a$ such that equation (1) has a solution $x(t)$ $\forall$ $t_0 - \delta < t < t_0 + \delta$.

   (ii) Prove that: if $f(t,x)$ is a Lipschitz continuous function in x, then the solution of equation (1) is unique.

2. Consider the following system of ODE:

$$X'(t) = AX(t), \quad \text{where} \quad A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}. \tag{2}$$

   (i) Find $e^{At} := I + \Sigma_{k=1}^{\infty} \frac{A^k t^k}{k!}$.

   (ii) Find the general solutions of equation (2).

3. (i) Assume that q_1 and q_2 are continuous functions satisfying $q_1(t) \leq q_2(t)$ on an interval $I \subset \mathbf{R}$. Let ϕ_1 and ϕ_2 be non-trivial solutions of equations

$$y'' + q_1(t)y = 0$$

and

$$y'' + q_2(t)y = 0$$

respectively, on I. Prove that: between any two consecutive zeroes α and β of ϕ_1, there exists at least one zero of ϕ_2 unless $\phi_1(t) \equiv \phi_2(t)$ on (α, β).

   (ii) Applying the result of above (i) to show that the equation

$$y''(x) + xy(x) = 0$$

has a non-trivial solution with infinitely many zeros on the inteerval $(1, \infty)$.

4. Consider the following two dimensional ODE

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy, \end{cases} \tag{3}$$

where a, b, c, d are real parameters. Discuss all the stability cases of equilibrium $(0,0)$ depending on these parameters.

5. Consider the nonlinear system:

$$\begin{cases} x' = y - x^3, \\ y' = -x - y^3. \end{cases} \tag{4}$$

Prove that: $(0,0)$ is the only equilibrium of (4), and $(0,0)$ is globally asymptotically stable. (**Hint** You can use the Lyapunov function $V(x,y) = x^2 + y^2$.)

1 *Consider the following initial value problem (IVP) on the real line* $\mathbb{R}$*:*

$$\begin{cases} \frac{dx}{dt} = 1 + x^2, \\ x(0) = x_0. \end{cases}$$

Answer the following questions:

(a) (11 points) Solve the differential equation using the method of separation of variables. Then apply the initial condition $x(0) = x_0$ *to obtain the particular solution* $x(t)$*.*

(b) (3 points)Let the maximal interval of existence of the solution $x(t)$ *be* $I = (\alpha, \beta)$*. Express* α *and* β *explicitly in terms of inverse trigonometric functions (in terms of* x_0*).*

(c) (3 points)Prove that as $t \to \alpha^+$ *and* $t \to \beta^-$*, we have* $|x(t)| \to \infty$*. What phenomenon does this describe?*

(d) (3 points)When $x_0 = 0$*, determine the maximal interval of existence. Explain why, even though the vector field* $f(x) = 1 + x^2$ *is smooth and well-defined on the entire real line* $\mathbb{R}$*, the solution cannot be extended to all time* $t \in (-\infty, \infty)$*.*

2 *Consider the one-parameter family of linear systems*

$$X' = \begin{pmatrix} a & \sqrt{2} + \frac{a}{2} \\ \sqrt{2} - \frac{a}{2} & 0 \end{pmatrix} X.$$

(a) (2 points) Let

$$A(a) = \begin{pmatrix} a & \sqrt{2} + \frac{a}{2} \\ \sqrt{2} - \frac{a}{2} & 0 \end{pmatrix}.$$

Compute the trace and determinant of the matrix $A(a)$*.*

(b) (3 points) Write down the characteristic equation

$$\det(\lambda I - A(a)) = 0.$$

(c) (3 points) Find the two eigenvalues $\lambda_1(a)$ *and* $\lambda_2(a)$*.*

(d) (2 points) Determine whether the eigenvalues can be complex. Is it possible for this system to undergo a Hopf bifurcation?

(e) (10 points) Discuss the bifurcations that occur along this parameter path as a varies.

Hint: *Bifurcations usually occur when an equilibrium changes from hyperbolic to nonhyperbolic. Find the parameter values a such that*

$$\lambda_1(a) = 0 \quad or \quad \lambda_2(a) = 0.$$

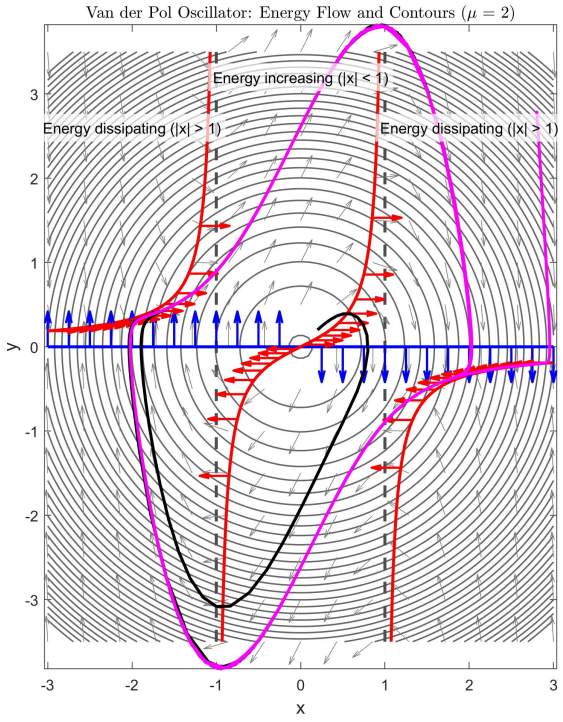


Figure 1: Phase portrait of the Van der Pol oscillator ($\mu = 2$), showing nullclines, normalized vector field, energy level curves, and representative inner and outer trajectories approaching the limit cycle.

3 *Consider the planar system illustrated in Fig.* (1)*:*

$$\begin{cases} x' = y, \\ y' = \mu(1 - x^2)y - x, \end{cases} \qquad \mu > 0.$$

(a) (3 point) Compute the Jacobian matrix at each equilibrium point and determine its type (e.g., stable node, unstable focus, saddle, etc.).

(b) (3 points) Define the function

$$V(x, y) = \frac{1}{2}(x^2 + y^2).$$

Compute $\dot{V}$ along trajectories of the system. Here, $\dot{V} = \frac{d}{dt}V(x(t), y(t))$.

(c) (2 points) Explain why

- *$\dot{V} > 0$ in the region $\{|x| < 1,\, y \neq 0\}$,*
- *$\dot{V} < 0$ in the region $\{|x| > 1,\, y \neq 0\}$.*

(d) (8 points) By applying Bendixson's Criterion, show that the system admits no closed curves contained wholly within the strip $|x| < 1$.

(e) (2 points) Deduce that there exists a bounded region in the plane that traps all trajectories starting in that region.

(f) (2 points) Using the Poincaré–Bendixson Theorem, conclude that the system admits at least one closed orbit (limit cycle).

4 *Consider the homogeneous linear system*

$$x'(t) = A(t)x(t),$$

where $A(t)$ is a 2×2 matrix whose entries are continuous on (r, s):

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}.$$

Let $x^{(1)}(t)$ and $x^{(2)}(t)$ be two solutions of the system on (r, s), where

$$x^{(1)}(t) = \begin{pmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{pmatrix}, \qquad x^{(2)}(t) = \begin{pmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{pmatrix}.$$

Define their Wronskian by

$$W(t) = \det \begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) \end{pmatrix}.$$

(a) (5 points) Show that

$$\frac{dW}{dt} = \begin{vmatrix} \dfrac{dx_1^{(1)}}{dt} & \dfrac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \dfrac{dx_2^{(1)}}{dt} & \dfrac{dx_2^{(2)}}{dt} \end{vmatrix}.$$

(b) (5 points) Use the differential equation $x' = A(t)x$ to prove that

$$\frac{dW}{dt} = \big(a_{11}(t) + a_{22}(t)\big)W(t).$$

(c) *(5 points) Solve the scalar differential equation obtained in part (b) and show that, for any fixed* $t_0 \in (r, s)$,

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \big(a_{11}(u) + a_{22}(u)\big)\, du \right).$$

(d) *(5 points) Use the formula in part (c) to prove that, on* (r, s), *the Wronskian either vanishes identically or never vanishes.*

5 ***Gronwall's Inequality.*** *Let* $u : [0, \alpha] \to \mathbb{R}$ *be a continuous nonnegative function. Assume that* $C \geq 0$ *and* $K \geq 0$ *are constants such that*

$$u(t) \leq C + \int_0^t Ku(s)\, ds$$

for all $t \in [0, \alpha]$. *Prove that*

$$u(t) \leq Ce^{Kt}$$

for all $t \in [0, \alpha]$. *(Hint:* $U(t) = C + \int_0^t Ku(s)\, ds$.*)*

1 *(20 points) Consider the initial value problem (IVP) on the real line* $\mathbb{R}$*:*

$$\begin{cases} \dfrac{dx}{dt} = 1 + x^2, \\ x(0) = 0. \end{cases} \tag{1}$$

(a) Solve the differential equation by separation of variables and use the initial condition $x(0) = 0$ *to find the particular solution* $x(t)$*.*

(b) Determine the maximal open interval of existence (α, β) *containing* $t = 0$ *for this solution.*

(c) Compute the limits $\lim_{t\to\alpha^+} x(t)$ *and* $\lim_{t\to\beta^-} x(t)$*.*

2 *(20 points) Consider the following linear system of ODEs*

$$X'(t) = AX(t), \quad \text{where } A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2}$$

(a) Compute the matrix exponential $e^{At} := I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$*. (Hint:* $e^{At} = e^{(2I+N)t}$*.)*

(b) Find the general solution of the system (2)*.*

3 *(20 points) Let* $p, q \geq 1$ *be odd positive integers, and let* $\alpha, \beta > 0$ *be constants. Consider the planar autonomous system*

$$\begin{cases} x' = y - \alpha x^p, \\ y' = -x - \beta y^q. \end{cases} \tag{3}$$

(a) Show that $(0, 0)$ *is the unique equilibrium point of the system* (3)*. (Hint:* $pq - 1 \geq 0$ *is even.)*

(b) Using the Lyapunov function $V(x, y) = \frac{1}{2}(x^2 + y^2)$*, prove that the origin* $(0, 0)$ *is globally asymptotically stable (GAS). Make sure to state why the radial unboundedness of* V *is required for global asymptotic stability.*

(Please refer to the Appendix.)

4 *(20 points)* ***Gronwall's Inequality.*** *Problem. Let $u, k : [0, \alpha] \to [0, \infty)$ be continuous functions, and let $C \geq 0$ be a constant. Suppose that*

$$u(t) \leq C + \int_0^t k(s)u(s)\, ds \quad \textit{for all } t \in [0, \alpha].$$

Prove that

$$u(t) \leq C \exp\left(\int_0^t k(s)\, ds\right) \quad \textit{for all } t \in [0, \alpha].$$

(Hint: Consider $U(t) = C + \int_0^t k(s)u(s)\, ds$ or an integrating factor.)

5 *(20 points) Consider the initial value problem in $\mathbb{R}^n$*

$$x'(t) = f(t, x(t)), \quad x(0) = x_0,$$

where $f : [0, \alpha] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and satisfies a generalized time-dependent Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq k(t)\|x - y\| \quad \textit{for all } t \in [0, \alpha],\ x, y \in \mathbb{R}^n,$$

with $k : [0, \alpha] \to [0, \infty)$ being a continuous function.

(a) Let $x_1(t)$ and $x_2(t)$ be solutions corresponding to initial conditions $x_1(0) = x_{0,1}$ and $x_2(0) = x_{0,2}$ respectively. Use the given ***Gronwall's Inequality*** *to prove that*

$$\|x_1(t) - x_2(t)\| \leq \|x_{0,1} - x_{0,2}\| \exp\left(\int_0^t k(s)\, ds\right) \quad \textit{for all } t \in [0, \alpha].$$

(b) Deduce that the initial value problem has at most one solution on $[0, \alpha]$.

Appendix: Global Asymptotic Stability Theorem

Consider the autonomous system

$$\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous.

Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the following three conditions

(1) **Positive Definiteness on $\mathbb{R}^n$:**

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \quad \forall x \neq 0.$$

(2) **Negative Definiteness of the Derivative:**

$$\dot{V}(x) = \nabla V(x) \cdot f(x) < 0, \quad \forall x \neq 0.$$

(3) **Radial Unboundedness (Properness):**

$$V(x) \to \infty \quad \text{as } \|x\| \to \infty.$$

Then, the origin $x = 0$ is **globally asymptotically stable (GAS)**; that is, for every initial state $x_0 \in \mathbb{R}^n$, the solution $x(t)$ satisfies

$$\lim_{t \to \infty} x(t) = 0.$$